Two-Commodity Flow is as Hard as Linear Programming

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Abstract

We give a nearly-linear time reduction that encodes any linear program as a 2-commodity flow problem with only a polylogarithmic blow-up in size. Our reduction applies to high-accuracy approximation algorithms and exact algorithms. Given an approximate solution to the 2-commodity flow problem, we can extract a solution to the linear program in linear time with only a polynomial factor increase in the error. This implies that any algorithm that solves the 2-commodity flow problem can solve linear programs in essentially the same time. Given a directed graph with edge capacities and two source-sink pairs, the goal of the 2-commodity flow problem is to maximize the sum of the flows routed between the two source-sink pairs subject to edge capacities and flow conservation. A 2-commodity flow problem can be formulated as a linear program, which can be solved to high accuracy in almost the current matrix multiplication time (Cohen-Lee-Song JACM'21). In this paper, we show that linear programs can be approximately solved, to high accuracy, using 2-commodity flow as well. As a corollary, if a 2-commodity flow problem can be approximately solved in time $O(|E|^c \operatorname{polylog}(U\,|E|\,\epsilon^{-1}))$, where E is the graph edge set, U is the ratio of maximum to minimum edge capacity, ϵ is the multiplicative error parameter, and c is a constant greater than or equal to 1, then a linear program with integer coefficients and feasible set radius r can be approximately solved in time $O(N^c \operatorname{polylog}((r+1)X\epsilon^{-1}))$, where N is the number of nonzeros and X is the largest magnitude of the coefficients. Thus a solver for 2-commodity flow with running time exponent $c < \omega$, where $\omega < 2.37286$ is the matrix multiplication constant, would improve the running time for solving sparse linear programs.

Our proof follows the outline of Itai's polynomial-time reduction of a linear program to a 2-commodity flow problem (JACM'78). Itai's reduction shows that exactly solving 2-commodity flow and exactly solving linear programming are polynomial-time equivalent. We improve Itai's reduction to preserve the sparsity of all the intermediate steps. In addition, we establish an error bound for approximately solving each intermediate problem in the reduction, and show that the accumulated error is polynomially bounded. We remark that our reduction does not run in strongly polynomial time and that it is open whether 2-commodity flow and linear programming are equivalent in strongly polynomial time.

1 Introduction

In this paper, we consider the very well-studied multi-commodity maximum flow problem. Many variants of multi-commodity flow problems exist. We consider one of the simplest directed variants, 2-commodity maximum through-put flow. Given a directed graph with edge capacities and two source-sink pairs, this problem requires us to maximize the sum of the flows routed between the two source-sink pairs, while satisfying capacity constraints and flow conservation at the remaining nodes. In the rest of the paper, we will simply refer to this as *the 2-commodity flow problem*. We abbreviate this problem as 2CF. Our goal is to relate the hardness of solving 2CF to that of solving linear programs (LPs). 2-commodity flow is easily expressed as a linear program, so it is clearly no harder than solving LPs. We show that the 2-commodity flow problem can encode a linear program with only a polylogarithmic blow-up in size. Our reduction runs in nearly-linear time. Given an approximate solution to the 2-commodity flow problem, we can recover, in linear time, an approximate solution to the linear program with only a polynomial factor increase in the error. This also implies that an exact solution to the flow problem yields an exact solution to the linear program.

Multi-commodity flow problems are extremely well-studied and have been the subject of numerous surveys [Ken78; AMO93; OMV00; BKV09; Wan18], in part because a large number of problems can be expressed as variants of multi-commodity flow. Our result shows that this is no accident - general linear programs can be expressed this way. Early in the study of these problems, before a polynomial-time algorithm for linear programming was known, it was shown that the *undirected* 2-commodity flow problem can be solved in polynomial time [Hu63]. In fact, it can be reduced to two undirected single commodity maximum flow problems. In contrast, directed 2-commodity flow problems were seemingly harder, despite the discovery of non-trivial algorithms for some special cases [Eva76; Eva78].

Alon Itai [Ita78] proved a polynomial-time reduction from linear programming to 2-commodity flow, before a polynomial-time algorithm for linear programming was known. For decades, the only major progress on solving multi-commodity flow came from improvements to general linear program solvers [Kha80; Kar84; Ren88; Vai89]. Then, Leighton et al. [Lei+95] showed that undirected capacitated k-commodity flow in a graph with m edges and n vertices can be approximately solved in $\widetilde{O}(kmn)$, albeit with a poor dependence on ϵ when completely routing all demands with $1+\epsilon$ congestion of the edges. This beats solve-times for linear programming in sparse graphs for small k, even with today's LP solvers that run in current matrix multiplication time, albeit with much worse error. This spurred a number of follow-up works with improvements for low-accuracy algorithms [GK07; Fle00; Mad10]. Later, breakthroughs in achieving almost- and nearly-linear time algorithms for undirected single-commodity maximum flow also lead to faster algorithms for undirected k-commodity flow [Kel+14; She13; Pen16], culminating in Sherman's introduction of area-convexity to build a $\widetilde{O}(mk\epsilon^{-1})$ time algorithm for approximate undirected k-commodity flow [She17].

Undirected multi-commodity flow has played a central role in approximation algorithms for NP-hard problems. It was core to the result of Leighton and Rao [LR89; LR99] which gave a breakthrough $O(\log n)$-approximation algorithm for sparsest cut and gave approximation algorithms for a host of other NP-complete problems. Building on this development, [Kle+90] showed that a related general concurrent flow problem can be used to approximately solve an even wider array of NP-complete problems, and more results in this vein have followed e.g. [LRS98]. An important development in the research on undirected multi-commodity flow for approximation algorithms was the characterization of the multi-commodity flow-cut gap [LR89; LR99; AR98; LLR95; CSW10], as well as the relationship of this phenomenon to graph metric embeddings [Gup+04]. Understanding of the undirected multi-commodity flow-cut gap has in turn played an important role in the devel-

opment of fast graph algorithms including algorithms for single commodity undirected maximum flow [Kel+14].

In contrast, the flow-cut gap for directed variants of multi-commodity flow and approximation algorithms for directed multi-cut are not as well-understood, but the directed multi-commodity flow-cut gap is known to be polynomial in contrast to the logarithmic gap of the undirected case [CK09; AAC07; CM16; SSS19].

Single commodity flow problems have been an area of tremendous success for the development of graph algorithms, with an era of algorithms deeply influenced by early results on maximum flow and minimum cut [FF56] and later the development of powerful combinatorial algorithms for maximum flow [Din70; ET75; GR98] with polynomially bounded edge capacities. Later, a breakthrough result on nearly-linear time for electrical flows by Spielman and Teng [ST04] lead to the *Laplacian paradigm*. A long line of work explored direct improvements and simplifications of this result [KMP10; KMP11; Kel+13; PS14; KS16; JS21]. This also motivated a new line of research on undirected maximum flow [Chr+11; LRS13; Kel+14; She13], which in turn lead to faster algorithms for directed maximum flow and minimum cost flow [Mad13; Mad16; LS20; KLS20; van+21; GLP21] building on powerful tools using mixed-ℓ_2, ℓ_p-norm minimizing flows [Kyn+19] and inverse-maintenance ideas [Che+20]. Certain developments are particularly relevant to our result: For a graph $G = (V, E)$ these works established high-accuracy algorithms with $\widetilde{O}(|E|)$ running time for computing electrical flow [ST04] and $O(|E|^{4/3})$ running time for unit capacity directed maximum flow [Mad13; KLS20], and $\widetilde{O}(\min(|E|^{1.497}, |E| + |V|^{1.5}))$ running time for directed maximum flow with general capacities [GLP21; van+21].

The many successes in developing high-accuracy algorithms for single-commodity flow problems highlight an important open question: Can multi-commodity flow be solved to high accuracy faster than general linear programs? Maximum flow in a sparse graph with $|E| = \widetilde{O}(|V|)$ edges (and polynomially bounded capacities) can be solved in time $\widetilde{O}(|V|^{1.497})$, thus one could hope to solve sparse multi-commodity flow problems in time faster than the $\widetilde{O}(|V|^{2.372\ldots})$ running time provided by LP solvers that run in current matrix multiplication time [CLS21]. However, our result shows that any improvement for sparse multi-commodity flow to high accuracy would directly translate to a faster algorithm for solving sparse linear programs to high accuracy, with only a polylogarithmic increase in running time.

Previous work by Kyng and Zhang [KZ20] had shown that fast algorithms for multi-commodity flow were unlikely to arise from combining interior point methods with special-purpose linear equation solvers. Concretely, they showed that the linear equations that arise in interior point methods for multi-commodity flow are as hard to solve as arbitrary linear equations. This ruled out algorithms following the pattern of the known fast algorithms for high-accuracy single-commodity flow problems. However, it left open the broader question that if some other family of algorithms could succeed. We now show that in general, a separation between multi-commodity flow and linear programming is not possible.

1.1 Previous works

Our paper follows the proof by Itai [Ita78] that linear programming is polynomial-time reducible to 2-commodity flow. However, it is also inspired by recent works on hardness for structured linear equations [KZ20] and packing/covering LPs [KWZ20], which focused on obtaining nearly-linear time reductions in somewhat related settings. These works in turn were motivated by the last decade's substantial progress on fine-grained complexity for a range of polynomial time solvable problems, e.g. see [WW18]. Also notable is the result by Musco et al. [Mus+19] on hardness for matrix spectrum approximation.

1.2 Our contributions

In this paper, we explore the hardness of 2-commodity maximum throughput flow, which for brevity we refer to as the 2-commodity flow problem or 2CF. We relate the difficulty of this problem to that of linear programming (LP). We develop a nearly-linear time, sparsity-preserving polynomial reduction from LP to 2CF, and we show that given an approximate 2CF solution, we can obtain an approximate LP solution with only polynomially larger error. More precisely, given an LP with N nonzero integer coefficients in the range $[-X, X]$ and polytope radius of R in ℓ_1 norm, we can obtain a 2CF with $|E| = O(N \log X)$ edges and integer capacities on the order $O(N^3 R X^{2.01})$ in time $O(N \log X)$. And if we want to solve an LP within error ϵ, then we can reduce it to solving a 2CF within $\Omega\left(\frac{\epsilon}{N^{10} R^3 X^{7.01}}\right)$ multiplicative error in congestion. This means that if 2CF can be solved within ϵ error in time $O\left(|E|^c \cdot \operatorname{poly}\log(1/\epsilon)\right)$, then LP can be solved within ϵ error in time $O\left(N^c \cdot \operatorname{poly}\log\left(\frac{NRX}{\epsilon}\right)\right)$.

We obtain our result by making several improvements to Itai's reduction from LP to 2CF. First of all, while Itai produced a 2CF with the number of edges in the order $O\left(N^2 \log^2 X\right)$, we show that an improved gadget can reduce this to $O\left(N \log X\right)$, so that the problem sparsity can be preserved. Itai used very large graph edge capacities that require $O\left((N \log X)^{1.01}\right)$ many bits *per edge*, as the capacities grew exponentially given an LP with polynomially bounded entries. We show that by making a natural assumption on the LP, namely that all feasible solutions have ℓ_1 norm at most R for some $R > 0$, we can ensure that capacities remain a polynomial function of the initial parameters N, R, and X. Hence each capacity can be represented using $\log(N)$ bits, assuming the parameters X and R are bounded by $\operatorname{poly}(N)$. In fact, if there exists a feasible solution $\boldsymbol{x}$ satisfying $\|\boldsymbol{x}\|_1 \leq R$, then we can add a constraint $\|\boldsymbol{x}\|_1 \leq R$ to the LP[1] so that in the new LP, all feasible solutions have ℓ_1 norm at most R. This only increases the number of nonzeros in the LP by at most a constant factor. It is common in LP solvers to consider polytopes with such a bounded radius R, [Ren88; CLS21].

Crucially, while Itai analyzed the chain of reductions under the case with exact solutions, we generalize the analysis to the case with approximate solutions by establishing an error analysis along the chain. We show that the error only grows polynomially during the reduction. More precisely, if a 2CF can be solved within ϵ error in congestion, then the LP can be solved within error $O(N^{10} R^3 X^{7.01} \epsilon)$. Moreover, to simplify our error analysis, we observe that additional structures can be established in many of Itai's reductions. For instance, we propose the notion of a *fixed flow network*, which consists of a subset of edges with equal lower and upper bound of capacity. It is a simplification of Itai's (l, u) network with general capacity (both lower and upper bounds on the amount of flow).

We remark that while we have carried out our error analysis assuming a 2CF solver with multiplicative error in congestion (or equivalently error in demand at the source and sink nodes), we believe that it should be straightforward to extend our analysis to 2CF solvers with additive error in demand at all nodes while still retaining polynomial blow-up in error.

1.3 Organization of the remaining paper

In Section 2, we give some general notations and problem definitions. These definitions include those problems that are involved in the reduction from LP to 2CF. In Section 3, we state our main theorem, and overview the proof that reduces an LP instance to a 2CF instance by a chain of efficient reductions. In Section 4, we provide proof details for all the steps along the chain. In each step, we describe a (nearly-)linear-time method of reducing a problem A to a problem B, and a

[1] Wlog, we can assume $\boldsymbol{x} \geq \mathbf{0}$. Then, $\|\boldsymbol{x}\|_1 = \sum_i \boldsymbol{x}(i)$ is linear.

linear-time method of mapping a solution of B to a solution of A. More importantly, we prove that the size of B is nearly linear in that of A, and an approximate solution to B can be mapped back to an approximate solution to A with a polynomial blow-up in error parameters. In Section 5, we prove the main theorem by putting all intermediate steps together.

2 Preliminaries

2.1 Notation

Matrices and vectors We use parentheses to denote entries of a matrix or a vector: Let $\boldsymbol{A}(i,j)$ denote the (i,j)th entry of a matrix $\boldsymbol{A}$, and let $\boldsymbol{x}(i)$ denote the ith entry of a vector $\boldsymbol{x}$. Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, we use $\boldsymbol{a}_i^\top$ to denote the ith row of a matrix $\boldsymbol{A}$ and $\mathrm{nnz}(\boldsymbol{A})$ to denote the number of nonzero entries of $\boldsymbol{A}$. Without loss of generality, we assume that $\mathrm{nnz}(\boldsymbol{A}) \geq \max\{m, n\}$. For any vector $\boldsymbol{x} \in \mathbb{R}^n$, we define $\|\boldsymbol{x}\|_{max} = \max_{i \in [n]} |\boldsymbol{x}(i)|$, $\|\boldsymbol{x}\|_1 = \sum_{i \in [n]} |\boldsymbol{x}(i)|$. For any matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, we define $\|\boldsymbol{A}\|_{max} = \max_{i,j} |\boldsymbol{A}(i,j)|$.

We define a function X that takes an arbitrary number of matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_{k_1}$, vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{k_2}$, and scalars $K_1, \ldots, K_{k_3}$ as arguments, and returns the maximum of $\|\cdot\|_{max}$ of all the arguments, i.e.,

$$\begin{aligned} &X(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_{k_1}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{k_2}, K_1, \ldots, K_{k_3}) \\ &\qquad = \max\left\{ \|\boldsymbol{A}_1\|_{\max}, \ldots, \|\boldsymbol{A}_{k_1}\|_{\max}, \|\boldsymbol{b}_1\|_{\max}, \ldots, \|\boldsymbol{b}_{k_2}\|_{\max}, |K_1|, \ldots, |K_{k_3}| \right\}. \end{aligned}$$

2.2 Problem Definitions

In this section, we formally define approximately solving a linear program and approximately solving a 2-commodity flow problem. In addition, we formally define the problems that we use in the reduction. These problems fall into two categories: one category is related to linear programming and linear equations, and the other is related to flow problems in graphs.

2.2.1 Linear Programming and Linear Equations with Positive Variables

For the convenience of our reduction, we define linear programming as a "decision" problem. We can solve the optimization problem $\max\{\boldsymbol{c}^\top \boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ by binary searching its optimal value via the decision problem.

Since we are interested in linear programs specified using finite precision coefficients, we assume we are given a linear program with integer coefficients. A linear program with rational coefficients can be converted to a linear program with integer coefficients by multiplying all coefficients with an appropriate common scaling factor.

Definition 2.1 (Linear programming (LP))**.** Given a matrix $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$, vectors $\boldsymbol{b} \in \mathbb{Z}^m$ and $\boldsymbol{c} \in \mathbb{Z}^n$, an integer K, and $R \geq \max\{1, \max\{\|\boldsymbol{x}\|_1 : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}\}$, we refer to the LP problem for $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R)$ as the problem of finding a vector $\boldsymbol{x} \in \mathbb{R}^n_{\geq 0}$ satisfying

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \text{ and } \boldsymbol{c}^\top \boldsymbol{x} \geq K$$

if such an $\boldsymbol{x}$ exists and returning "infeasible" otherwise.

We also define an approximate version of solving linear programs.

Definition 2.2 (LP Approximate Problem (LPA))**.** An LPA instance is given by an LP instance $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R)$ and an error parameter $\epsilon \in [0, 1]$, which we collect in a tuple $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R, \epsilon)$. We say an algorithm solves the LPA problem, if, given any LPA instance, it returns a vector $\boldsymbol{x} \geq \boldsymbol{0}$ such that

$$\begin{aligned} \boldsymbol{c}^\top \boldsymbol{x} &\geq K - \epsilon \\ \boldsymbol{A}\boldsymbol{x} &\leq \boldsymbol{b} + \epsilon \boldsymbol{1} \end{aligned}$$

where $\boldsymbol{1}$ is the all-1 vector, or it correctly declares that the associated LP instance is infeasible.

Remark. Note that our definition of LPA does not require the algorithm to provide a certificate of infeasibility – but our notion of an *algorithm* for LPA requires the algorithm never incorrectly asserts infeasibility. Also note that when the LP instance is infeasible, the algorithm is still allowed to return an approximately feasible solution, if it finds one.

We use the same approach to defining all our approximate decision problems.

We will reduce a linear program to linear equations with nonnegative variables (LEN) and linear equations with nonnegative variables and small integral coefficients (k-LEN). We will define the exact version of these two problems below, and define their approximate version in Section 4.

Definition 2.3 (Linear Equations with Nonnegative Variables (LEN))**.** Given $\boldsymbol{A} \in \mathbb{Z}^{m \times n}, \boldsymbol{b} \in \mathbb{Z}^m$, and $R \geq \max\{1, \max\{\|\boldsymbol{x}\|_1 : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}\}$, we refer to the LEN problem for $(\boldsymbol{A}, \boldsymbol{b}, R)$ as the problem of finding a vector $\boldsymbol{x} \in \mathbb{R}^n_{\geq 0}$ satisfying $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ if such an $\boldsymbol{x}$ exists and returning "infeasible" otherwise.

Definition 2.4 (k-LEN (k-LEN))**.** The k-LEN problem is an LEN problem $(\boldsymbol{A}, \boldsymbol{b}, R)$ where the entries of $\boldsymbol{A}$ are integers in $[-k, k]$ for some given $k \in \mathbb{Z}_+$, which we collect in a tuple $(\boldsymbol{A}, \boldsymbol{b}, R, k)$

2.2.2 Flow Problems

A *flow network* is a directed graph $G = (V, E)$, where V is the set of vertices and $E \subset V \times V$ is the set of edges, together with a vector of edge capacities $\boldsymbol{u} \in \mathbb{Z}^{|E|}_{>0}$ that upper bound the amount of flow passing each edge. A *2-commodity flow network* is a flow network together with two source-sink pairs $s_i, t_i \in V$ for each commodity $i \in \{1, 2\}$.

Given a 2-commodity flow network $(G = (V, E), \boldsymbol{u}, s_1, t_1, s_2, t_2)$, a *feasible 2-commodity flow* is a pair of flows $\boldsymbol{f}_1, \boldsymbol{f}_2 \in \mathbb{R}^{|E|}_{\geq 0}$ that satisfies

1. capacity constraint: $\boldsymbol{f}_1(e) + \boldsymbol{f}_2(e) \leq \boldsymbol{u}(e),\ \forall e \in E$, and
2. conservation of flows: $\sum_{u:(u,v)\in E} \boldsymbol{f}_i(u, v) = \sum_{w:(v,w)\in E} \boldsymbol{f}_i(v, w),\ \forall i \in \{1, 2\}, v \in V \setminus \{s_i, t_i\}$.

For each commodity flow $\boldsymbol{f}_i$, we let $F_i = \sum_{v:(s_i,v)\in E} \boldsymbol{f}_i(s_i, v)$ be the amount of flow $\boldsymbol{f}_i$ routed from s_i to t_i.

Similar to the definition of LP, we define 2-commodity flow problem as a decision problem. We can solve a decision problem by solving the corresponding optimization problem.

Definition 2.5 (2-Commodity Flow Problem (2CF))**.** Given a 2-commodity flow network $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2)$ together with $R \geq 0$, we refer to the 2CF problem for $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2, R)$ as the problem of finding a feasible 2-commodity flow $\boldsymbol{f}_1, \boldsymbol{f}_2$ satisfying

$$F_1 + F_2 \geq R$$

if such flows exist and returning "infeasible" otherwise.

Definition 2.6 (2CF Approximate Problem (2CFA))**.** A 2CFA instance is given by a 2CF instance $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2, R)$ and error parameters $\epsilon, \epsilon' \in [0, 1]$, which we collect in a tuple $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2, R, \epsilon, \epsilon')$. We say an algorithm solves the 2CFA problem, if, given any 2CFA instance, it returns a pair of flows $\boldsymbol{f}_1, \boldsymbol{f}_2 \geq 0$ that satisfies the conservation of flows at every vertex other than s_1, t_1, s_2, t_2 and

$$\boldsymbol{f}_1(e) + \boldsymbol{f}_2(e) \leq (1 + \epsilon)\boldsymbol{u}_e, \ \forall e \in E \tag{1}$$
$$F_1 + F_2 \geq (1 - \epsilon')R \tag{2}$$

or it correctly declares that the associated 2CF instance is infeasible. We refer to the error in (1) as error in congestion, and the error in (2) as error in demand.

Remark. Our definition of 2CFA works with multiplicative error in demand, thus there is no error in demand for vertices $v \in V \backslash \{s_1, s_2, t_1, t_2\}$ because their net demand is zero by the conservation of flows.

Our definition of 2CFA allows both error in congestion and error in demand. The following lemma shows that we can transfer a 2-commodity flow with both error in congestion and error in demand to a 2-commodity flow with only error in congestion, by scaling the flows. We defer the proof of Lemma 2.7 to Appendix A.

Lemma 2.7 (2CFA simplification)**.** *Given a solution to a* 2CFA *instance* $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2, R, \epsilon, \epsilon')$*, we can reduce it, in linear time, to a solution to* 2CFA $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2, R, \tilde{\epsilon}, 0)$*, where* $\tilde{\epsilon}$ *is the error in congestion and satisfies*

$$\tilde{\epsilon} = \frac{\epsilon + \epsilon'}{1 - \epsilon'}.$$

In the rest of the paper, we will only consider error in congestion for 2CFA and represent a 2CFA instance by a tuple $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2, R, \tilde{\epsilon})$, where $\tilde{\epsilon}$ is the error in congestion.

To reduce LP to 2CF, we need a sequence of variants of flow problems. Again, here we only define the exact version of these problems. Later, we will define their approximate version in Section 4.

Definition 2.8 (2-Commodity Flow with Required Flow Amount (2CFR))**.** Given a 2-commodity flow network $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2)$ together with $R_1, R_2 \geq 0$, we refer to the 2CFR for $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2, R_1, R_2)$ as the problem of finding a feasible 2-commodity flow $\boldsymbol{f}_1, \boldsymbol{f}_2$ satisfying

$$F_1 \geq R_1, \quad F_2 \geq R_2$$

if such flows exist and returning "infeasible" otherwise.

Definition 2.9 (Fixed flow constraints)**.** Given a set $F \subseteq E$ in a 2-commodity flow network, we say the flows $\boldsymbol{f}_1, \boldsymbol{f}_2$ satisfy *fixed flow constraints on* F if

$$\boldsymbol{f}_1(e) + \boldsymbol{f}_2(e) = \boldsymbol{u}(e), \ \forall e \in F.$$

Similarly, given a set $F \subseteq E$ in a 1-commodity flow network, we say the flow $\boldsymbol{f}$ satisfies *fixed flow constraints on* F if

$$\boldsymbol{f}(e) = \boldsymbol{u}(e), \ \forall e \in F.$$

Definition 2.10 (2-Commodity Fixed Flow Problem (2CFF))**.** Given a 2-commodity flow network $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2)$ together with a subset of edges $F \subseteq E$, we refer to the 2CFF problem for the tuple $(G, F, \boldsymbol{u}, s_1, t_1, s_2, t_2)$ as the problem of finding a feasible 2-commodity flow $\boldsymbol{f}_1, \boldsymbol{f}_2 \geq \boldsymbol{0}$ which also satisfies the fixed flow constraints on F if such flows exist and returning "infeasible" otherwise.

Definition 2.11 (Selective Fixed Flow Problem (SFF))**.** Given a 2-commodity network $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2)$ together with three edge sets $F, S_1, S_2 \subseteq E$, we refer to the SFF problem for $(G, F, S_1, S_2, \boldsymbol{u}, s_1, t_1, s_2, t_2)$ as the problem of finding a feasible 2-commodity flow $\boldsymbol{f}_1, \boldsymbol{f}_2 \geq \boldsymbol{0}$ such that for each $i \in \{1, 2\}$, flow $\boldsymbol{f}_i(e) > 0$ only if $e \in S_i$, and $\boldsymbol{f}_1, \boldsymbol{f}_2$ satisfy the fixed flow constraints on F, if such flows exist, and returning "infeasible" otherwise.

Definition 2.12 (Fixed Homologous Flow Problem (FHF))**.** Given a flow network with a single source-sink pair $(G, \boldsymbol{u}, s, t)$ together with a collection of disjoint subsets of edges $\mathcal{H} = \{H_1, \ldots, H_h\}$ and a subset of edges $F \subseteq E$ such that F is disjoint from all the sets in $\mathcal{H}$, we refer to the FHF problem for $(G, F, \mathcal{H}, \boldsymbol{u}, s, t)$ as the problem of finding a feasible flow $\boldsymbol{f} \geq \boldsymbol{0}$ such that

$$\boldsymbol{f}(e_1) = \boldsymbol{f}(e_2), \quad \forall e_1, e_2 \in H_k, 1 \leq k \leq h,$$

and $\boldsymbol{f}$ satisfies the fixed flow constraints on F, if such flows exist, and returning "infeasible" otherwise.

Definition 2.13 (Fixed Pair Homologous Flow Problem (FPHF))**.** The FPHF is an FHF problem $(G, F, \mathcal{H}, \boldsymbol{u}, s, t)$ where every set in $\mathcal{H}$ has size 2.

3 Main results

Theorem 3.1. *Given an* LPA *instance* $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R, \epsilon^{lp})$ *where* $\boldsymbol{A} \in \mathbb{Z}^{m \times n}, \boldsymbol{b} \in Z^m, \boldsymbol{c} \in \mathbb{Z}^n, K \in \mathbb{Z}$ *and* $\boldsymbol{A}$ *has* $\operatorname{nnz}(\boldsymbol{A})$ *nonzero entries, we can reduce it to a* 2CFA *instance* $(G = (V, E), \boldsymbol{u}, s_1, t_1, s_2, t_2, R^{2cf}, \epsilon^{2cf})$*, in time* $O(\operatorname{nnz}(\boldsymbol{A}) \log X)$ *where* $X = X(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K)$*, such that*

$$\begin{aligned}
&|V|, |E| = O(\operatorname{nnz}(\boldsymbol{A}) \log X), \\
&\|\boldsymbol{u}\|_{max}, R^{2cf} = O(\operatorname{nnz}^3(\boldsymbol{A}) R X^2 \log^2 X), \\
&\epsilon^{2cf} = \Omega\left(\frac{1}{\operatorname{nnz}^{10}(\boldsymbol{A}) R^3 X^7 \log^7 X}\right) \epsilon^{lp},
\end{aligned}$$

and if the LP *instance* $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R)$ *has a solution, then the* 2CF *instance* $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2, R^{2cf})$ *has a solution. Furthermore, if* $\boldsymbol{f}^{2cf}$ *is a solution to the* 2CFA *instance, then in time* $O(\operatorname{nnz}(\boldsymbol{A}) \log X)$*, we can compute a solution* $\boldsymbol{x}$ *to the* LPA *instance.*

Our main theorem immediately implies the following corollary.

Corollary 3.2. *If we can solve any* 2CFA *instance* $(G = (V, E), \boldsymbol{u}, s_1, t_1, s_2, t_2, R^{2cf}, \epsilon)$ *in time* $O\left(|E|^c \operatorname{poly}\log\left(\frac{\|\boldsymbol{u}\|_1}{\epsilon}\right)\right)$ *for some small constant* $c \geq 1$*, then we can solve any* LPA *instance* $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R, \epsilon)$ *in time* $O\left(\operatorname{nnz}^c(\boldsymbol{A}) \operatorname{poly}\log\left(\frac{\operatorname{nnz}(\boldsymbol{A}) R X(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K)}{\epsilon}\right)\right)$.

3.1 Overview of our proof

We give a summary of notations used in the reduction from LP(A) to 2CF(A), as shown in Table 1.

In this section, we will explain how to reduce an LP instance to a 2-commodity flow (2CF) instance by a chain of efficient reductions between different problems. In each step, we reduce a decision problem A to a decision problem B; we guarantee that (1) the reduction runs in nearly linear time[2], (2) the size of B is nearly linear in that of A, and (3) that A is feasible implies that B

[2]Linear in the size of problem A, poly-logarithmic in the maximum magnitude of all the numbers that describe A, the feasible set radius, and the inverse of the error parameter if an approximate solution is allowed.

Table 1: A summary of notations used in the reduction from LP(A) to 2CF(A). The column "Input" and "Output" are shared for both exact and approximate problems. The column "Error" is only for approximate problems.

Exact problem	Input	Output	Approximate problem	Error [3]
LP (Def. 2.1)	$\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R$	$\boldsymbol{x}$	LPA (Def. 2.2)	ϵ^{lp}
LEN (Def. 2.3)	$\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{R}$	$\tilde{\boldsymbol{x}}$	LENA (Def. 4.2)	ϵ^{le}
2-LEN (Def. 2.4)	$\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{R}, 2$	$\bar{\boldsymbol{x}}$	2-LENA (Def. 4.5)	ϵ^{2le}
1-LEN (Def. 2.4)	$\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}, \hat{R}, 1$	$\hat{\boldsymbol{x}}$	1-LENA (Def. 4.5)	ϵ^{1le}
FHF (Def. 2.12)	$G^h, F^h, \mathcal{H}^h = \{H_1, \cdots, H_h\}, \boldsymbol{u}^h, s, t$	$\boldsymbol{f}^h$	FHFA (Def. 4.10)	$\epsilon^h_l, \epsilon^h_u, \epsilon^h_d, \epsilon^h_h$
FPHF (Def. 2.13)	$G^p, F^p, \mathcal{H}^p = \{H_1, \cdots, H_p\}, \boldsymbol{u}^p, s, t$	$\boldsymbol{f}^p$	FPHFA (Def. 4.13)	$\epsilon^p_l, \epsilon^p_u, \epsilon^p_d, \epsilon^p_h$
SFF (Def. 2.11)	$G^s, F^s, S_1, S_2, \boldsymbol{u}^s, s_1, t_1, s_2, t_2$	$\boldsymbol{f}^s$	SFFA (Def. 4.16)	$\epsilon^s_l, \epsilon^s_u, \epsilon^s_{d1}, \epsilon^s_{d2}, \epsilon^s_{t1}, \epsilon^s_{t2}$
2CFF (Def. 2.10)	$G^f, F^f, \boldsymbol{u}^f, s_1, t_1, s_2, t_2$	$\boldsymbol{f}^f$	2CFFA (Def. 4.19)	$\epsilon^f_l, \epsilon^f_u, \epsilon^f_{d1}, \epsilon^f_{d2}$
2CFR (Def. 2.8)	$G^r, \boldsymbol{u}^r, \bar{s}_1, \bar{t}_1, \bar{s}_2, \bar{t}_2, R_1, R_2$	$\boldsymbol{f}^r$	2CFRA (Def. 4.22)	$\epsilon^r, \epsilon^r_1, \epsilon^r_2$
2CF (Def. 2.5)	$G^{2cf}, \boldsymbol{u}^{2cf}, \bar{\bar{s}}_1, \bar{t}_1, \bar{\bar{s}}_2, \bar{t}_2, R^{2cf}$	$\boldsymbol{f}^{2cf}$	2CFA (Def. 2.6)	ϵ^{2cf}

is feasible, and an approximate solution to B can be turned to an approximate solution to A with only a polynomial blow-up in error parameters, in linear time.

We follow the outline of Itai's reduction [Ita78]. Itai first reduced an LP instance to a 1-LEN instance (linear equations with nonnegative variables and ± 1 coefficients). A 1-LEN instance can be represented by a single-commodity flow problem subject to additional homologous constraints and fixed flow constraints (FHF). Then, Itai dropped these additional constraints step by step, via introducing a second commodity of flow and imposing lower bound requirements on the total amount of flows routed between the source-sink pairs. However, in the worst case, Itai's reduction from 1-LEN to FHF enlarges the problem size quadratically and is thus inefficient. One of our main contributions is to improve this step so that the sparsity is preserved along the reduction chain.

Our second main contribution is an upper bound on the errors accumulated during the process of mapping an approximate solution to the 2CFA instance to an approximate solution to the LPA instance. We show that the error only grows by polynomial factors. Itai only considered exact solutions between these two instances, and showed that exactly solving LP and 2CF are (polynomially) (up to polynomial factors) equivalent. Our analysis on approximate solutions also implies that approximately solving LPA and 2CFA are (up to polylogarithmic factors) equivalent for algorithms whose running time scales as $\text{polylog}(1/\epsilon)$ when computing an ϵ error solution.

We will explain the reductions based on the exact versions of the problems. At the end of this section, we will discuss some intuitions of behind our error analysis.

3.1.1 Reducing Linear Programming to Linear Equations with Nonnegative Variables and ± 1 Coefficients

Given an LP instance $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R)$ where $R \geq \max\{1, \max\{\|x\|_1 : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}\}$, we want to compute a vector $\boldsymbol{x} \geq \boldsymbol{0}$ satisfying

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{c}^\top \boldsymbol{x} \geq K$$

or to correctly declare infeasible. We introduce slack variables $\boldsymbol{s}, \alpha \geq \boldsymbol{0}$ and turn the above inequalities to equalities:

$$\boldsymbol{A}\boldsymbol{x} + \boldsymbol{s} = \boldsymbol{b}, \boldsymbol{c}^\top \boldsymbol{x} - \alpha = K$$

[3] Error parameters will be defined in Section 4.

which is an LEN instance $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{R})$. Comparing to Itai's proof, we need to track two additional parameters: R, the polytope radius, and X, the maximum magnitude of all numbers appearing in the problem instance specification.

We then reduce the LEN instance to linear equations with ± 2 coefficients (2-LEN) by bitwise decomposition. For each bit, we need to introduce a carry term, represented as a difference between two nonnegative variables. In contrast to Itai's reduction, we impose an upper bound for each carry variable. We show that this upper bound does not change problem feasibility and it guarantees the polytope radius only increases polynomially. Next, we reduce the 2-LEN instance to a 1-LEN instance by replacing each variable with coefficient ± 2 by two new equal-valued variables.

All the above three reduction steps run in nearly linear time, and the problem sizes increase nearly linearly.

3.1.2 Reducing Linear Equations with Nonnegative Variables and ± 1 Coefficients to Fixed Homologous Flow Problem

One of our main contributions is a linear-time reduction from 1-LEN to FHF (single-commodity fixed homologous flow problem). Our reduction is similar to Itai's reduction, but more efficient.

Itai observed that a linear equation $\boldsymbol{a}^\top \boldsymbol{x} = b$ with ± 1 coefficients can be represented as a fixed homologous flow network G. G has a source vertex s, a sink vertex t, and two additional vertices J^+ and J^-. Each variable $\boldsymbol{x}(i)$ corresponds to an edge: There is an edge from s to J^+ if $\boldsymbol{a}(i) = 1$, and an edge from s to J^- if $\boldsymbol{a}(i) = -1$. The amount of flow passing this edge corresponds to the value of $\boldsymbol{x}(i)$. The difference between the total amount of flow entering J^+ and that entering J^- equals to $\boldsymbol{a}^\top \boldsymbol{x}$. To force $\boldsymbol{a}^\top \boldsymbol{x} = b$, we add two edges e_1, e_2 from J^+ to t and one edge e_3 from J^- to t [4]; we require e_1 and e_3 to be homologous and require e_2 to be a fixed flow with value b.

The above construction can be generalized to a system of linear equations (see Figure 1 in Section 4.4). Specifically, we create a gadget as above for each equation i, and then glue all the source (sink) vertices for each equation together as the source (sink, respectively) of the graph. To force the consistency of the variable values, we require the edges corresponding to the same variable in different equations to be homologous. The number of the vertices is linear in the number of equations; the number of the edges and the total size of the homologous sets are both linear in the number of nonzero coefficients of the linear equation system.

3.1.3 Dropping the Homologous and Fixed Flow Constraints

To reduce FHF to 2CF (2-commodity flow problem), we need to drop the homologous and fixed flow constraints. The reduction has three main steps.

Reducing FHF to SFF. Given an FHF instance, we can reduce it to a fixed homologous flow instance in which each homologous edge set has size 2 (FPHF). To drop the homologous requirement in FPHF, we introduce a second commodity of flow with source-sink pair (s_2, t_2), and for each edge, we carefully select the type(s) of flow that can pass through this edge. Specifically, given two homologous edges (v, w) and (y, z), we construct a constant-sized gadget (see Figure 4 in Section 4.6): We introduce new vertices vw, vw', yz, yz', construct a directed path $P : s_2 \to vw \to vw' \to yz \to yz' \to t_2$, and add edges $(v, vw), (vw', w)$ and $(y, yz), (yz', z)$. Now, there is a directed path $P_{vw} : v \to vw \to vw' \to w$ and a directed path $P_{yz} : y \to yz \to yz' \to z$. Paths P and P_{vw} (P_{yz}) share an edge $e_{vw} = (vw, vw')$ ($e_{yz} = (yz, yz')$, respectively). We select e_{vw} and e_{yz} for both flow $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$, select the rest of the edges along P for only $\boldsymbol{f}_2$, and select the rest of the edges along

[4] We remark that e_1, e_2 are multi-edges, but after the next reduction step, we will get a simple graph.

P_{vw}, P_{yz} for only $\boldsymbol{f}_1$. By this construction, in this gadget, we have $\boldsymbol{f}_2(e_{vw}) = \boldsymbol{f}_2(e_{yz})$ being the amount of flow routed in P, $\boldsymbol{f}_1(e_{vw})$ and $\boldsymbol{f}_1(e_{yz})$ being the amount of flow routed in P_{vw} and P_{yz}, respectively. Next, we choose e_{vw} and e_{yz} to be fixed flow edges with equal capacity; this guarantees the same amount of $\boldsymbol{f}_1$ is routed through P_{vw} and P_{yz}. The new graph is an SFF instance.

Reducing SFF to 2CFF. Next, we will drop the selective requirement of the SFF instance. For each edge (x, y) selected for flow i, we construct a constant-sized gadget (see Figure 5 in Section 4.7): We introduce two vertices xy, xy', construct a direct path $s_i \to xy' \to xy \to t_i$, and add edge (x, xy) and (xy', y). This gadget simulates a directed path from x to y for flow $\boldsymbol{f}_i$, and guarantees no directed path from x to y for flow $\boldsymbol{f}_{\bar{i}}$ so that $\boldsymbol{f}_{\bar{i}}$ cannot be routed from x to y. We get a 2CFF instance.

Reducing 2CFF to 2CF. It remains to drop the fixed flow constraints. The gadget we will use is similar to that used in the last step. We first introduce new sources $\bar{s}_1, \bar{s}_2$ and sinks $\bar{t}_1, \bar{t}_2$. Then, for each edge (x, y) with capacity u, we construct a constant-sized gadget (see Figure 6 in Section 4.8). We introduce two vertices xy, xy', add edges $(\bar{s}_1, xy'), (\bar{s}_2, xy'), (xy, \bar{t}_1), (xy, \bar{t}_2), (xy', xy)$, and $(x, xy), (xy', y)$. This simulates a directed path from x to y that both flow $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ can pass through. We let (xy', xy) have capacity u if (x, y) is a fixed flow edge and $2u$ otherwise; we let all the other edges have capacity u. Assume all the edges incident to the sources and the sinks are saturated, then the total amount of flows routed from x to y in this gadget must be u if (x, y) is a fixed flow edge and no larger than u otherwise. Moreover, since the original sources and sinks are no longer sources and sinks now, we have to satisfy the conservation of flows at these vertices. For each $i \in \{1, 2\}$, we create a similar gadget involving $\bar{s}_i, \bar{t}_i$ to simulate a directed path from t_i to s_i (the original sink and source), and let the edges incident to $\bar{s}_i, \bar{t}_i$ have capacity M, the sum of all the edge capacities in the 2CFF instance. This gadget guarantees that assuming the edges incident to $\bar{s}_i$ and $\bar{t}_i$ are saturated, the amount of flow routed from t_i to s_i through this gadget can be any number at most M. To force the above edge-saturation assumptions to hold, we require the amount of flow $\boldsymbol{f}_i$ routed from $\bar{s}_i$ to $\bar{t}_i$ to be no less than $2M$ for each $i \in \{1, 2\}$.

Now, this instance is close to a 2CF instance except that we require a lower bound for each flow value instead of a lower bound for the sum of two flow values. To handle this, we introduce new sources $\bar{\bar{s}}_1, \bar{\bar{s}}_2$ and for each $i \in \{1, 2\}$, we add an edge $(\bar{\bar{s}}_i, \bar{s}_i)$ with capacity $2M$, the lower bound required for the value of $\boldsymbol{f}_i$.

One can check that in each reduction step, the reduction time is nearly linear and the problem size increases nearly linearly. In addition, given a solution to the 2CF instance, one can construct a solution to the LP instance in nearly linear time.

We also establish an error bound for mapping an approximate solution to 2CFA to an approximate solution to LPA. Although 2CFA only has error in congestion, when we map a 2CFA solution back, we introduce different types of errors such as error in requirement, demand, selective types, and homology. This is because, in the forward direction, a reduction step may map an edge to multiple edges; while in the backward direction, we have to map the flows passing through a gadget including multiple edges to a flow passing a single edge. A key observation is that the errors at each vertex or edge accumulate in an additive way. More precisely, the increment of errors can be upper bounded by a weighted sum of the errors at the vertices or on the edges nearby, and these weights are polynomially bounded. So, the final error only increases polynomially.

Contents

4 Proof details

The chain of reductions from LP(A) to 2CF(A) consists of nine steps. In each step, we analyze the reduction from some problem A to some problem B in both the exact case and the approximate case. In the exact case, we start with describing a nearly-linear-time method of reducing A to B, and a nearly-linear-time method of mapping a solution of B back to a solution of A. Then, we prove the correctness of the reduction method in the forward direction, that is if A has a solution then B has a solution. In addition, we provide a fine-grained analysis of the size of B given the size of A.

In the approximate case, we formally define approximately solving each problem in the first place, specifying bounds on various different types of error. We always use the same reduction method from problem A to problem B in the exact and approximate cases. Thus the conclusion that problem B has a feasible solution when problem A has a feasible solution also applies in the approximate case.

We also always use a solution map back for the approximate case that agrees with the exact case when there is no error. Crucially, we conduct an error analysis. That is given an approximate solution to B and its error parameters (by abusing notations, we use ϵ^B to denote), we map it back to an approximate solution to A and measure its error τ^A with respect to ϵ^B. In other words, we can reduce an approximate version of A with error parameters $\epsilon^A \geq \tau^A$ to an approximate version of B with error parameters ϵ^B. Note that the correctness of the reduction method in the backward direction for the exact case follows from the approximate case analysis by setting all error parameters to zero, which completes the proof of correctness.

4.1 LP(A) to LEN(A)

4.1.1 LP to LEN

We show the reduction from an LP instance $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R)$ to an LEN instance $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{R})$. The LP instance has the following form:

$$\begin{aligned} \boldsymbol{c}^\top \boldsymbol{x} &\geq K \\ \boldsymbol{A}\boldsymbol{x} &\leq \boldsymbol{b} \\ \boldsymbol{x} &\geq \boldsymbol{0} \end{aligned} \tag{3}$$

To reduce it to an LEN instance, we introduce slack variables α and $\boldsymbol{s}$:

$$\begin{aligned} \begin{pmatrix} \boldsymbol{c}^\top & \boldsymbol{0} & -1 \\ \boldsymbol{A} & \boldsymbol{I} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{s} \\ \alpha \end{pmatrix} &= \begin{pmatrix} K \\ \boldsymbol{b} \end{pmatrix} \\ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{s} \\ \alpha \end{pmatrix} &\geq \boldsymbol{0} \end{aligned} \tag{4}$$

Setting

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{c}^\top & \boldsymbol{0} & -1 \\ \boldsymbol{A} & \boldsymbol{I} & \boldsymbol{0} \end{pmatrix}, \qquad \tilde{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{s} \\ \alpha \end{pmatrix}, \qquad \tilde{\boldsymbol{b}} = \begin{pmatrix} K \\ \boldsymbol{b} \end{pmatrix},$$

we get an LEN instance $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{R})$ where $\tilde{R} = \max\{1, \max\{\|\tilde{\boldsymbol{x}}\|_1 : \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{x}} \geq \boldsymbol{0}\}\}$.

If an LEN solver returns $\tilde{\boldsymbol{x}} = (\boldsymbol{x}^\top, \boldsymbol{s}^\top, \alpha)^\top$ for the LEN instance $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{R})$, then we return $\boldsymbol{x}$ for the LP instance $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R)$; if the LEN solver returns "infeasible" for the LEN instance, then we return "infeasible" for the LP instance.

Lemma 4.1 (LP to LEN)**.** *Given an* LP *instance* $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R)$ *where* $\boldsymbol{A} \in \mathbb{Z}^{m\times n}, \boldsymbol{b} \in \mathbb{Z}^m, \boldsymbol{c} \in \mathbb{Z}^n, K \in \mathbb{Z}$*, we can construct, in* $O(\mathrm{nnz}(\boldsymbol{A}))$ *time, an* LEN *instance* $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{R})$ *where* $\tilde{\boldsymbol{A}} \in \mathbb{Z}^{\tilde{m}\times\tilde{n}}, \tilde{\boldsymbol{b}} \in \mathbb{Z}^{\tilde{m}}$ *such that*

$$\tilde{n} = n + m + 1, \quad \tilde{m} = m + 1, \quad \mathrm{nnz}(\tilde{\boldsymbol{A}}) \le 4\,\mathrm{nnz}(\boldsymbol{A}),$$
$$\tilde{R} = 5mRX(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K), \quad X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}) = X(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K) \ge 1,$$

and if the LP *instance has a solution, then the* LEN *instance has a solution.*

Proof. Based on the reduction method described above, if $\boldsymbol{x}$ is a solution to the LP instance as shown in Eq. (3), we can derive a solution $\tilde{\boldsymbol{x}} = (\boldsymbol{x}^\top, \boldsymbol{s}^\top, \alpha)^\top$ to the LEN instance as shown in Eq. (4), by setting

$$\boldsymbol{s} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}, \qquad \alpha = \boldsymbol{c}^\top\boldsymbol{x} - K.$$

Thus, if the LP instance has a solution, then the LEN instance has a solution.

Given the size of the LP instance with n variables, m linear constraints, and $\mathrm{nnz}(\boldsymbol{A})$ nonzero entries, we observe the size of the reduced LEN instance as following:

1. $\tilde{n}$ variables, where
$$\tilde{n} = n + m + 1.$$

2. $\tilde{m}$ linear constraints, where
$$\tilde{m} = m + 1.$$

3. $\mathrm{nnz}(\tilde{\boldsymbol{A}})$ nonzeros, where
$$\mathrm{nnz}(\tilde{\boldsymbol{A}}) = \mathrm{nnz}(\boldsymbol{A}) + \mathrm{nnz}(\boldsymbol{c}) + m + 1 \le 4\,\mathrm{nnz}(\boldsymbol{A}), \tag{5}$$
where we use $\mathrm{nnz}(\boldsymbol{A}) \ge m, n \ge 1$, and $\mathrm{nnz}(\boldsymbol{c}) \le n$.

4. $\tilde{R} = \max\{1, \max\{\|\tilde{\boldsymbol{x}}\|_1 : \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{x}} \ge \boldsymbol{0}\}\}$, the radius of polytope in ℓ_1 norm. Our goal is to upper bound $\|\tilde{\boldsymbol{x}}\|_1$ for every feasible solution to the LEN instance. By definition and the triangle inequality,
$$\|\tilde{\boldsymbol{x}}\|_1 \le \|\boldsymbol{x}\|_1 + \alpha + \|\boldsymbol{s}\|_1.$$
Note $\|\boldsymbol{x}\|_1 \le R$, the polytope radius in the LP instance. In addition,
$$\alpha = \boldsymbol{c}^\top\boldsymbol{x} - K \le \|\boldsymbol{c}\|_{max}\|\boldsymbol{x}\|_1 + |K| \le \|\boldsymbol{c}\|_{max} R + |K|,$$
and
$$\|\boldsymbol{s}\|_1 = \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_1 \le \|\boldsymbol{A}\boldsymbol{x}\|_1 + \|\boldsymbol{b}\|_1 \le m\|\boldsymbol{A}\|_{max}\|\boldsymbol{x}\|_1 + \|\boldsymbol{b}\|_1 \le m(\|\boldsymbol{A}\|_{max} R + \|\boldsymbol{b}\|_{max}).$$
Therefore, we have
$$\begin{aligned}\|\tilde{\boldsymbol{x}}\|_1 &\le R + |K| + \|\boldsymbol{c}\|_{max} R + m(\|\boldsymbol{A}\|_{max} R + \|\boldsymbol{b}\|_{max}) \\ &\le mR(1 + |K| + \|\boldsymbol{c}\|_{max} + \|\boldsymbol{A}\|_{max} + \|\boldsymbol{b}\|_{max}) && \text{Because } R \ge 1 \\ &\le 5mRX(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K).\end{aligned}$$
Hence, it suffices to set
$$\tilde{R} = 5mRX(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K).$$

5. $X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}) = X(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K)$ because

$$\left\|\tilde{\boldsymbol{A}}\right\|_{max} = \max\left\{\|\boldsymbol{A}\|_{max}, \|\boldsymbol{c}\|_{max}, 1\right\} = \max\left\{\|\boldsymbol{A}\|_{max}, \|\boldsymbol{c}\|_{max}\right\},$$

and

$$\left\|\tilde{\boldsymbol{b}}\right\|_{max} = \max\left\{|K|, \|\boldsymbol{b}\|_{max}\right\}.$$

To estimate the reduction time, as it takes $O\left(\mathrm{nnz}(\tilde{\boldsymbol{A}})\right)$ time to construct $\tilde{\boldsymbol{A}}$, and $O\left(\mathrm{nnz}(\tilde{\boldsymbol{b}})\right)$ time to construct $\tilde{\boldsymbol{b}}$, thus the reduction takes time

$$\begin{aligned} O\left(\mathrm{nnz}(\tilde{\boldsymbol{A}}) + \mathrm{nnz}(\tilde{\boldsymbol{b}})\right) &= O\left(\mathrm{nnz}(\tilde{\boldsymbol{A}})\right) && \text{Because } \mathrm{nnz}(\tilde{\boldsymbol{b}}) \le m \le \mathrm{nnz}(\tilde{\boldsymbol{A}}) \\ &= O\left(\mathrm{nnz}(\boldsymbol{A})\right). && \text{By Eq. (5)} \end{aligned}$$

□

4.1.2 LPA to LENA

The above lemma shows the reduction between exactly solving an LP instance and exactly solving an LEN instance. Next, we generalize the case with exact solutions to the case that allows approximate solutions. First of all, we give a definition of the approximate version of LEN.

Definition 4.2 (LEN Approximate Problem (LENA))**.** an LENA instance is given by an LEN instance $(\boldsymbol{A}, \boldsymbol{b}, R)$ as in Definition 2.3 and an error parameter $\epsilon \in [0, 1]$, which we collect in a tuple $(\boldsymbol{A}, \boldsymbol{b}, R, \epsilon)$. We say an algorithm solves the LENA problem, if, given any LENA instance, it returns a vector $\boldsymbol{x} \ge \boldsymbol{0}$ such that

$$|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}| \le \epsilon \boldsymbol{1},$$

where $|\cdot|$ is entrywise absolute value and $\boldsymbol{1}$ is the all-1 vector, or it correctly declares that the associated LEN instance is infeasible.

Remark. We use an additive error in LENA so that it is consistent with the additive error used in LP. Note this is different from a multiplicative error, which is also commonly used in solving systems of linear equations.

We use the same reduction method in the exact case to reduce an LPA instance to an LENA instance. Furthermore, if an LENA solver returns $\tilde{\boldsymbol{x}} = (\boldsymbol{x}^\top, \boldsymbol{s}^\top, \alpha)^\top$ for the LENA instance $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{R}, \epsilon^{le})$, then we return $\boldsymbol{x}$ for the LPA instance $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R, \epsilon^{lp})$; if the LENA solver returns "infeasible" for the LENA instance, then we return "infeasible" for the LPA instance.

Lemma 4.3 (LPA to LENA)**.** *Given an* LPA *instance* $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R, \epsilon^{lp})$ *where* $\boldsymbol{A} \in \mathbb{Z}^{m\times n}, \boldsymbol{b} \in \mathbb{Z}^m, \boldsymbol{c} \in \mathbb{Z}^n, K \in \mathbb{Z}$*, we can construct, in* $O(\mathrm{nnz}(\boldsymbol{A}))$ *time, an* LENA *instance* $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{R}, \epsilon^{le})$ *where* $\tilde{\boldsymbol{A}} \in \mathbb{Z}^{\tilde{m}\times\tilde{n}}, \tilde{\boldsymbol{b}} \in \mathbb{Z}^{\tilde{m}}$ *such that*

$$\begin{gathered} \tilde{n} = n + m + 1, \quad \tilde{m} = m + 1, \quad \mathrm{nnz}(\tilde{\boldsymbol{A}}) \le 4\,\mathrm{nnz}(\boldsymbol{A}), \\ \tilde{R} = 5mRX(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K), \quad X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}) = X(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K), \\ \epsilon^{le} = \epsilon^{lp}. \end{gathered}$$

If the LP *instance* $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, R)$ *has a solution, then the* LEN *instance* $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{R})$ *has a solution. Furthermore, if* $\tilde{\boldsymbol{x}}$ *is a solution to the* LENA *instance, then in time* $O(n)$*, we can compute a solution* $\boldsymbol{x}$ *to the* LPA *instance.*

Proof. Since we use the same reduction method in the exact case to the approximate case, the conclusions in Lemma 4.1 also apply here, including the reduction time, problem size, and that the LEN instance has a feasible solution when the LP instance has one. It remains to show the solution mapping time, as well as how the problem error changes by mapping an approximate solution to the LENA instance back to an approximate solution to the LPA instance.

Based on the solution mapping method described above, given a solution $\tilde{\boldsymbol{x}}$, we discard those entries of $\boldsymbol{s}$ and α, and map back trivially for those entries of $\boldsymbol{x}$. As it takes constant time to set the value of each entry of $\boldsymbol{x}$ by mapping back trivially, and the size of $\boldsymbol{x}$ is n, thus the solution mapping takes $O(n)$ time.

Now, we conduct an error analysis. If $\tilde{\boldsymbol{x}} = (\boldsymbol{x}^\top, \boldsymbol{s}^\top, \alpha)^\top$ is a solution to the LENA instance, then by Definition 4.2, $\tilde{\boldsymbol{x}}$ satisfies

$$\begin{aligned} \left| \boldsymbol{c}^\top \boldsymbol{x} - \alpha - K \right| &\leq \epsilon^{le}, \\ |\boldsymbol{A}\boldsymbol{x} + \boldsymbol{s} - \boldsymbol{b}| &\leq \epsilon^{le} \boldsymbol{1}. \end{aligned}$$

Taking one direction of the absolution value, we obtain

$$\begin{aligned} \boldsymbol{c}^\top \boldsymbol{x} &\geq \alpha + K - \epsilon^{le} \geq K - \epsilon^{le}, \\ \boldsymbol{A}\boldsymbol{x} &\leq -\boldsymbol{s} + \boldsymbol{b} + \epsilon^{le} \boldsymbol{1} \leq \boldsymbol{b} + \epsilon^{le} \boldsymbol{1}, \end{aligned}$$

As we set in the reduction that $\epsilon^{le} = \epsilon^{lp}$, then we have

$$\begin{aligned} \boldsymbol{c}^\top \boldsymbol{x} &\geq K - \epsilon^{lp}, \\ \boldsymbol{A}\boldsymbol{x} &\leq \boldsymbol{b} + \epsilon^{lp} \boldsymbol{1}, \end{aligned}$$

which indicates that $\boldsymbol{x}$ is a solution to the LPA instance by Definition 2.2. □

4.2 LEN(A) to 2-LEN(A)

4.2.1 LEN to 2-LEN

We show the reduction from an LEN instance $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{R})$ to a 2-LEN instance $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{R}, 2)$. The LEN instance has the following form:

$$\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}},$$

where $\tilde{\boldsymbol{A}} \in \mathbb{Z}^{\tilde{m} \times \tilde{n}}, \tilde{\boldsymbol{b}} \in \mathbb{Z}^{\tilde{m}}$. To reduce it to a 2-LEN instance $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{R}, 2)$ in which the coefficients of $\bar{\boldsymbol{A}}$ are in $\{\pm 1, \pm 2\}$, we do *bitwise decomposition*. Algorithm 1 describes how to obtain a binary representation of an integer. The algorithm takes $z \in \mathbb{Z}$ as an input and output a list L consisting of all the powers such that $z = s(z) \sum_{l \in L} 2^l$ where $s(z)$ is the sign of z. For example, $z = -5$, then $L = \{2, 0\}$ and $s(z) = -1$.

We will reduce each linear equation in LEN to a linear equation in 2-LEN. For an arbitrary linear equation q of LEN: $\tilde{\boldsymbol{a}}_q^\top \tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}(q), q \in [\tilde{m}]$, we describe the reduction by the following 4 steps.

1. We run Algorithm 1 for each nonzero entry of $\tilde{\boldsymbol{a}}_q$ and $\tilde{\boldsymbol{b}}(q)$ so that each nonzero entry has a binary representation. To simplify notations, we denote the sign of $\tilde{\boldsymbol{a}}_q(i)$ as s_q^i and the list returned by BINARYREPRESENTATION$(\tilde{\boldsymbol{a}}_q(i))$ as L_q^i, and the sign of $\tilde{\boldsymbol{b}}(q)$ as s_q and the list

Algorithm 1: BINARYREPRESENTATION

Input: $z \in \mathbb{Z}$

Output: L is a list of powers of 2 such that $z = s(z) \sum_{l \in L} 2^l$, where $s(z)$ returns the sign of z.

```
r ← |z|;
L ← [];
for r > 0 do
    L.append(⌊log_2 r⌋);
    r ← r − 2^⌊log_2 r⌋;
end
```

returned by BINARYREPRESENTATION($\tilde{\boldsymbol{b}}(q)$) as L_q. Thus, the qth linear equation of LEN can be rewritten as

$$\sum_{i \in [\tilde{n}]} \underbrace{\left(s_q^i \sum_{l \in L_q^i} 2^l \right)}_{\tilde{\boldsymbol{a}}_q(i)} \tilde{\boldsymbol{x}}(i) = \underbrace{s_q \sum_{l \in L_q} 2^l}_{\tilde{\boldsymbol{b}}(q)}. \tag{6}$$

2. Letting N_q denote the maximum element of L_q and $L_q^i, i \in [\tilde{n}]$, i.e.,

$$N_q = \left\lfloor \log_2 \max \left\{ \|\tilde{\boldsymbol{a}}_q\|_{max}, \left| \tilde{\boldsymbol{b}}(q) \right| \right\} \right\rfloor,$$

then we can rearrange the left hand side of Eq. (6) by gathering those terms located at the same bit (i.e., with the same weight of power of 2), and obtain

$$\sum_{l=0}^{N_q} \left(\sum_{i \in [\tilde{n}]} \mathbb{1}_{[l \in L_q^i]} \cdot s_q^i \tilde{\boldsymbol{x}}(i) \right) 2^l = s_q \sum_{l \in L_q} 2^l, \tag{7}$$

where $\mathbb{1}$ is an indicator function such that

$$\mathbb{1}_{[l \in L_q^i]} = \begin{cases} 1 & \text{if } l \in L_q^i, \\ 0 & \text{otherwise.} \end{cases}$$

3. We decompose Eq. (7) into $N_q + 1$ linear equations such that each one representing a bit, by matching its left hand side and right hand side of Eq. (7) that are located at the same bit. We will also introduce a carry term for each bit, which passes the carry from the equation corresponding to that bit to the equation corresponding to the next bit. Without carry terms, the new system may be infeasible even if the old one is. Moreover, since a carry can be any real number, we represent each carry as a difference of two nonnegative variables $\boldsymbol{c}_q(i) - \boldsymbol{d}_q(i), \boldsymbol{c}_q(i), \boldsymbol{d}_q(i) \geq 0$. Starting from the lowest bit, the following are $N_q + 1$ linear

equations after decomposition.

$$\begin{aligned}
\sum_{i\in[\tilde{n}]} \mathbb{1}_{[0\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i) - 2[\boldsymbol{c}_q(0)-\boldsymbol{d}_q(0)] &= s_q\mathbb{1}_{[0\in L_q]}\\
\sum_{i\in[\tilde{n}]} \mathbb{1}_{[1\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i) + [\boldsymbol{c}_q(0)-\boldsymbol{d}_q(0)] - 2[\boldsymbol{c}_q(1)-\boldsymbol{d}_q(1)] &= s_q\mathbb{1}_{[1\in L_q]}\\
&\vdots\\
\sum_{i\in[\tilde{n}]} \mathbb{1}_{[N_q\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i) + [\boldsymbol{c}_q(N_q-1)-\boldsymbol{d}_q(N_q-1)] &= s_q\mathbb{1}_{[N_q\in L_q]}
\end{aligned} \tag{8}$$

4. We add an additional constraint for each carry variable $\boldsymbol{c}_q(i), \boldsymbol{d}_q(i)$[5]:

$$\boldsymbol{c}_q(i) \le 2X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\tilde{R},\ \ \boldsymbol{d}_q(i) \le 2X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\tilde{R}. \tag{9}$$

These constraints guarantee that the polytope radius of the reduced 2-LEN instance cannot be too large [6]. In our proofs, we will show that these additional constraints do not affect the problem feasibility. We then add slack variables $\boldsymbol{s}_q^c(i), \boldsymbol{s}_q^d(i) \ge 0$ for each carry term and turn Eq. (9) to

$$\boldsymbol{c}_q(i) + \boldsymbol{s}_q^c(i) = 2X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\tilde{R},\quad \boldsymbol{d}_q(i) + \boldsymbol{s}_q^d(i) = 2X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\tilde{R},\qquad 0\le i\le N_q-1. \tag{10}$$

Repeating the above process for $\tilde{m}$ times, we get a 2-LEN instance $(\bar{\boldsymbol{A}},\bar{\boldsymbol{b}},\bar{R},2)$, where $\bar{\boldsymbol{A}}$ is the coefficient matrix, $\bar{\boldsymbol{b}}$ is the right hand side vector, and $\bar{R}$ is the polytope radius.

If a 2-LEN solver returns $\bar{\boldsymbol{x}} = (\tilde{\boldsymbol{x}}^\top, \boldsymbol{c}^\top, \boldsymbol{d}^\top, \boldsymbol{s}^{c\top}, \boldsymbol{s}^{d\top})^\top$ for the 2-LEN instance $(\bar{\boldsymbol{A}},\bar{\boldsymbol{b}},\bar{R},2)$, then we return $\tilde{\boldsymbol{x}}$ for the LEN instance $(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}},\tilde{R})$; if the 2-LEN solver returns "infeasible" for the 2-LEN instance, then we return "infeasible" for the LEN instance.

Lemma 4.4 (LEN to 2-LEN)**.** *Given an* LEN *instance* $(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}},\tilde{R})$ *where* $\tilde{\boldsymbol{A}}\in\mathbb{Z}^{\tilde{m}\times\tilde{n}}, \tilde{\boldsymbol{b}}\in\mathbb{Z}^{\tilde{m}}$*, we can construct, in* $O\left(\mathrm{nnz}(\tilde{\boldsymbol{A}})\log X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\right)$ *time, a* 2-LEN *instance* $(\bar{\boldsymbol{A}},\bar{\boldsymbol{b}},\bar{R},2)$ *where* $\bar{\boldsymbol{A}}\in\mathbb{Z}^{\bar{m}\times\bar{n}}, \bar{\boldsymbol{b}}\in\mathbb{Z}^{\bar{m}}$ *such that*

$$\bar{n}\le\tilde{n}+4\tilde{m}\left(1+\log X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\right),\quad \bar{m}\le 3\tilde{m}\left(1+\log X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\right),$$

$$\mathrm{nnz}(\bar{\boldsymbol{A}})\le 17\,\mathrm{nnz}(\tilde{\boldsymbol{A}})\left(1+\log X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\right),\quad \bar{R}=8\tilde{m}\tilde{R}X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\left(1+\log X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\right),$$

$$X(\bar{\boldsymbol{A}},\bar{\boldsymbol{b}}) = 2X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\tilde{R},$$

and if the LEN *instance has a solution, then the* 2-LEN *instance has a solution.*

Proof. Based on the reduction method described above, from any solution $\tilde{\boldsymbol{x}}$ to the LEN instance such that $\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}$, we can derive a solution $\bar{\boldsymbol{x}} = (\tilde{\boldsymbol{x}}^\top, \boldsymbol{c}^\top, \boldsymbol{d}^\top, \boldsymbol{s}^{c\top}, \boldsymbol{s}^{d\top})^\top$ to the 2-LEN instance. Concretely, for any linear equation q in LEN, $q\in[\tilde{m}]$, with its decomposed equations as shown in Eq. (8), we can set the value of $\boldsymbol{c}_q, \boldsymbol{d}_q$ from the highest bit as

$$\boldsymbol{c}_q(N_q-1) = \max\left\{0,\quad s_q\mathbb{1}_{[N_q\in L_q]} - \sum_{i\in[\tilde{n}]}\mathbb{1}_{[N_q\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i)\right\},$$

[5]We remark that Itai's reduction does not have these upper bound on carry variables. We need these constraints in our reduction to guarantee that the polytope radius is always well bounded.

[6]Note that without these additional constraints the radius of polytope can be unbounded.

$$\boldsymbol{d}_q(N_q-1)=\max\left\{0,\quad \sum_{i\in[\tilde{n}]}\mathbb{1}_{[N_q\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i)-s_q\mathbb{1}_{[N_q\in L_q]}\right\}.$$

And then using backward substitution, we can set the value for the rest entries of $\boldsymbol{c}_q$ and $\boldsymbol{d}_q$ similarly.

$$\begin{aligned}\boldsymbol{c}_q(N_q-2)&=\max\left\{0,\quad s_q\mathbb{1}_{[(N_q-1)\in L_q]}+2[\boldsymbol{c}_q(N_q-1)-\boldsymbol{d}_q(N_q-1)]-\sum_{i\in[\tilde{n}]}\mathbb{1}_{[(N_q-1)\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i)\right\}\\&=\max\left\{0,\quad \sum_{l=\{N_q-1,N_q\}}\left(s_q\mathbb{1}_{[l\in L_q]}-\sum_{i\in[\tilde{n}]}\mathbb{1}_{[l\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i)\right)2^{l-(N_q-1)}\right\},\\ \boldsymbol{d}_q(N_q-2)&=\max\left\{0,\quad \sum_{l=\{N_q-1,N_q\}}\left(\sum_{i\in[\tilde{n}]}\mathbb{1}_{[l\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i)-s_q\mathbb{1}_{[l\in L_q]}\right)2^{l-(N_q-1)}\right\};\\ &\vdots\\ \boldsymbol{c}_q(0)&=\max\left\{0,\quad s_q\mathbb{1}_{[1\in L_q]}+2[\boldsymbol{c}_q(1)-\boldsymbol{d}_q(1)]-\sum_{i\in[\tilde{n}]}\mathbb{1}_{[1\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i)\right\}\\&=\max\left\{0,\quad \sum_{l=1}^{N_q}\left(s_q\mathbb{1}_{[l\in L_q]}-\sum_{i\in[\tilde{n}]}\mathbb{1}_{[l\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i)\right)2^{l-1}\right\},\\ \boldsymbol{d}_q(0)&=\max\left\{0,\quad \sum_{l=1}^{N_q}\left(\sum_{i\in[\tilde{n}]}\mathbb{1}_{[l\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i)-s_q\mathbb{1}_{[l\in L_q]}\right)2^{l-1}\right\}.\end{aligned}$$

Substituting $\boldsymbol{c}_q(0),\boldsymbol{d}_q(0)$ back to the equation of the lowest bit, we will get Eq. (7), the rearranged binary representation of equation q: $\tilde{\boldsymbol{a}}_q^\top\tilde{\boldsymbol{x}}=\tilde{\boldsymbol{b}}(q)$. By our setting of the carry terms, we have

$$\begin{aligned}\boldsymbol{c}_q(i),\boldsymbol{d}_q(i)&\le\left|\sum_{l=i+1}^{N_q}\left(\sum_{i\in[\tilde{n}]}\mathbb{1}_{[l\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i)-s_q\mathbb{1}_{[l\in L_q]}\right)2^{l-i-1}\right|\\&\le 2^{N_q}(\|\tilde{\boldsymbol{x}}\|_1+1)\le 2\max\left\{\left\|\tilde{\boldsymbol{A}}\right\|_{\max},\left\|\tilde{\boldsymbol{b}}\right\|_{\max}\right\}\tilde{R}=2X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\tilde{R}.\end{aligned}$$

By setting slacks

$$\boldsymbol{s}_q^c(i)=2X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\tilde{R}-\boldsymbol{c}_q(i),\quad \boldsymbol{s}_q^d(i)=2X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\tilde{R}-\boldsymbol{d}_q(i),$$

we satisfy the equations (10) in the 2-LEN instance. Repeating the above process for all $q\in[\tilde{m}]$, we get a feasible solution $\bar{\boldsymbol{x}}=(\tilde{\boldsymbol{x}}^\top,\boldsymbol{c}^\top,\boldsymbol{d}^\top,\boldsymbol{s}^{c\top},\boldsymbol{s}^{d\top})^\top$ to the 2-LEN instance.

Now, we track the change of problem size after reduction. Based on the reduction method, each linear equation in LEN can be decomposed into at most N linear equations, where

$$N=1+\max_{q\in[\tilde{m}]}N_q=1+\left\lfloor\log_2\left\|\tilde{\boldsymbol{A}}\right\|_{max}\right\rfloor\le 1+\left\lfloor\log X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\right\rfloor.\tag{11}$$

Thus, given an LEN instance with $\tilde{n}$ variables, $\tilde{m}$ linear equations, and $\mathrm{nnz}(\tilde{\boldsymbol{A}})$ nonzero entries, we compute the size of the reduced 2-LEN instance as follows.

1. $\bar{n}$ variables. First, all $\tilde{n}$ variables in LEN are maintained. Then, for each of the $\tilde{m}$ equations in LEN, it is decomposed into at most N equations, where at most a pair of carry variables are introduced for each newly added equation. Finally, we introduce a slack variable for each carry variable. Thus, we have

$$\bar{n} \leq \tilde{n} + 2\tilde{m}N + 2\tilde{m}N \leq \tilde{n} + 4\tilde{m}\left(1 + \log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right).$$

2. $\bar{m}$ linear constraints. First, each equation in LEN is decomposed into at most N equations in 2-LEN. Next, we add a new constraint for each carry variable. Thus, we have

$$\bar{m} \leq \tilde{m}N + 2\tilde{m}N \leq 3\tilde{m}\left(1 + \log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right). \tag{12}$$

3. $\operatorname{nnz}(\bar{\boldsymbol{A}})$ nonzeros. To bound it, first, for each nonzero entry in $\tilde{\boldsymbol{A}}$, it will be decomposed into at most N bits, thus becomes at most N nonzero entries in $\bar{\boldsymbol{A}}$. Then, each equation in 2-LEN involves at most 4 carry variables. Furthermore, there are $2\tilde{m}N$ new constraints for carry variables, and each constraint involves a carry variable and a slack variable. In total, we have

$$\begin{aligned}
\operatorname{nnz}(\bar{\boldsymbol{A}}) &\leq \operatorname{nnz}(\tilde{\boldsymbol{A}})N + 4\bar{m} + 4\tilde{m}N \\
&\overset{(1)}{\leq} \operatorname{nnz}(\tilde{\boldsymbol{A}})\left(1 + \log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right) + 12\tilde{m}\left(1 + \log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right) + 4\tilde{m}\left(1 + \log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right) \\
&\overset{(2)}{\leq} 17\operatorname{nnz}(\tilde{\boldsymbol{A}})\left(1 + \log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right),
\end{aligned} \tag{13}$$

where in step (1), we utilize Eq. (12) for the bound of $\bar{m}$; and in step (2), we use $\tilde{m} \leq \operatorname{nnz}(\tilde{\boldsymbol{A}})$.

4. $\bar{R}$, the radius of the polytope in ℓ_1 norm. We want to upper bound $\bar{\boldsymbol{x}}_1$ for every feasible solution to the 2-LEN instance. By definition and the triangle inequality,

$$\|\tilde{\boldsymbol{x}}\|_1 \leq \|\tilde{\boldsymbol{x}}\|_1 + \|\boldsymbol{c}\|_1 + \|\boldsymbol{d}\|_1 + \|\boldsymbol{s}^c\|_1 + \left\|\boldsymbol{s}^d\right\|_1$$

Note $\|\tilde{\boldsymbol{x}}\|_1 \leq \tilde{R}$ and the maximum magnitude of the entries of $\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{s}^c, \boldsymbol{s}^d$ is at most $2X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\tilde{R}$ by Eq. (10). Also note the dimensions of $\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{s}^c, \boldsymbol{s}^d$ are $\tilde{m}N$. Thus,

$$\|\tilde{\boldsymbol{x}}\|_1 \leq \tilde{R} + 2X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\tilde{R} \cdot 4\tilde{m}N \leq 8\tilde{m}\tilde{R}X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\left(1 + \log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right).$$

Hence, it suffices to set

$$\bar{R} = 8\tilde{m}\tilde{R}X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\left(1 + \log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right).$$

5. $X(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}) = 2X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\tilde{R}$ because by construction,

$$\left\|\bar{\boldsymbol{A}}\right\|_{max} = 2, \qquad \left\|\bar{\boldsymbol{b}}\right\|_{max} = 2X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\tilde{R} \geq 2.$$

To estimate the reduction time, it is noticed that it takes $O(N)$ time to run Algorithm 1 for each nonzero entry of $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{b}}$. And there are at most $\operatorname{nnz}(\tilde{\boldsymbol{A}}) + \operatorname{nnz}(\tilde{\boldsymbol{b}}) = O\left(\operatorname{nnz}(\tilde{\boldsymbol{A}})\right)$ entries to be decomposed. In addition, it takes $O\left(\operatorname{nnz}(\bar{\boldsymbol{A}}) + \operatorname{nnz}(\bar{\boldsymbol{b}})\right) = O\left(\operatorname{nnz}(\bar{\boldsymbol{A}})\right)$ to construct $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{b}}$. By Eq. (13), performing such a reduction takes time in total

$$O\left(N\operatorname{nnz}(\tilde{\boldsymbol{A}}) + \operatorname{nnz}(\bar{\boldsymbol{A}})\right) = O\left(\operatorname{nnz}(\tilde{\boldsymbol{A}})\log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right).$$

□

4.2.2 LENA to 2-LENA

The above lemma shows the reduction between exactly solving an LEN instance and exactly solving a 2-LEN instance. Next, we generalize the case with exact solutions to the case that allows approximate solutions. First of all, we give a definition of the approximate version of 2-LEN.

Definition 4.5 (k-LEN Approximate Problem (k-LENA))**.** A k-LENA instance is given by a k-LEN instance $(\boldsymbol{A}, \boldsymbol{b}, R, k)$ as in Definition 2.4 and an error parameter $\epsilon \in [0, 1]$, which we collect in a tuple $(\boldsymbol{A}, \boldsymbol{b}, R, k, \epsilon)$. We say an algorithm solves the k-LENA problem, if, given any k-LENA instance, it returns a vector $\boldsymbol{x} \geq \boldsymbol{0}$ such that

$$|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}| \leq \epsilon \boldsymbol{1},$$

where $|\cdot|$ is entrywise absolute value and $\boldsymbol{1}$ is the all-1 vector, or it correctly declares that the associated k-LEN instance is infeasible.

We can use the same reduction method in the exact case to reduce an LENA instance to a 2-LENA instance. Furthermore, if a 2-LENA solver returns $\bar{\boldsymbol{x}} = (\tilde{\boldsymbol{x}}^\top, \boldsymbol{c}^\top, \boldsymbol{d}^\top, \boldsymbol{s}^{c\top}, \boldsymbol{s}^{d\top})^\top$ for the 2-LENA instance $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{R}, 2, \epsilon^{2le})$, then we return $\tilde{\boldsymbol{x}}$ for the LENA instance $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{R}, \epsilon^{le})$; if the 2-LENA solver returns "infeasible" for the 2-LENA instance, then we return "infeasible" for the LENA instance.

Lemma 4.6 (LENA to 2-LENA)**.** *Given an* LENA *instance* $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{R}, \epsilon^{le})$*, where* $\tilde{\boldsymbol{A}} \in \mathbb{Z}^{\tilde{m}\times\tilde{n}}, \tilde{\boldsymbol{b}} \in \mathbb{Z}^{\tilde{m}}$*, we can construct, in* $O\left(\operatorname{nnz}(\tilde{\boldsymbol{A}}) \log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right)$ *time, a* 2-LENA *instance* $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{R}, 2, \epsilon^{2le})$ *where* $\bar{\boldsymbol{A}} \in \mathbb{Z}^{\bar{m}\times\bar{n}}, \bar{\boldsymbol{b}} \in \mathbb{Z}^{\bar{m}}$ *such that*

$$\bar{n} \leq \tilde{n}+4\tilde{m}\left(1+\log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right), \quad \bar{m} \leq 3\tilde{m}\left(1+\log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right), \quad \operatorname{nnz}(\bar{\boldsymbol{A}}) \leq 17\operatorname{nnz}(\tilde{\boldsymbol{A}})\left(1+\log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right),$$

$$\bar{R} = 8\tilde{m}\tilde{R}X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\left(1+\log X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\right), \quad X(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}) = 2X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})\tilde{R},$$

$$\epsilon^{2le} = \frac{\epsilon^{le}}{2X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}})}.$$

If the LEN *instance* $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{R})$ *has a solution, then the* 2-LEN *instance* $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{R}, 2)$ *has a solution. Furthermore, if* $\bar{\boldsymbol{x}}$ *is a solution to the* 2-LENA *instance, then in time* $O(\tilde{n})$*, we can compute a solution* $\tilde{\boldsymbol{x}}$ *to the* LENA *instance.*

Proof. Since we use the same reduction method in the exact case to the approximate case, the conclusions in Lemma 4.4 also apply here, including the reduction time, problem size, and that 2-LEN has a feasible solution when LEN has one. It remains to show the solution mapping time, as well as how the problem error changes by mapping an approximate solution to 2-LENA back to an approximate solution to LENA.

Based on the solution mapping method described above, given a solution $\bar{\boldsymbol{x}}$, we discard those entries of carry variables and slack variables, and map back trivially for those entries of $\tilde{\boldsymbol{x}}$. As it takes constant time to set the value of each entry of $\tilde{\boldsymbol{x}}$ by mapping back trivially, and the size of $\tilde{\boldsymbol{x}}$ is $\tilde{n}$, thus the solution mapping takes $O(\tilde{n})$ time.

Now, we conduct an error analysis. By Definition 4.5, the error of each linear equation in 2-LENA can be bounded by ϵ^{2le}. In particular, the equation in the 2-LENA instance that corresponds to the highest bit of the qth equation in the LENA instance satisfies

$$\left|\sum_{i\in[\tilde{n}]} \mathbb{1}_{[N_q\in L_q^i]} \cdot s_q^i \tilde{\boldsymbol{x}}(i) + [\boldsymbol{c}_q(N_q-1) - \boldsymbol{d}_q(N_q-1)] - s_q \mathbb{1}_{[N_q\in L_q]}\right| \leq \epsilon^{2le},$$

which can be rearranged as

$$-\epsilon^{2le}-[\boldsymbol{c}_q(N_q-1)-\boldsymbol{d}_q(N_q-1)] \le \sum_{i\in[\tilde{n}]} \mathbb{1}_{[N_q\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i)-s_q\mathbb{1}_{[N_q\in L_q]} \le \epsilon^{2le}-[\boldsymbol{c}_q(N_q-1)-\boldsymbol{d}_q(N_q-1)]. \quad (14)$$

For the second highest bit, we have

$$\begin{aligned}-\epsilon^{2le}+2[\boldsymbol{c}_q(N_q-1)-\boldsymbol{d}_q(N_q-1)]-[\boldsymbol{c}_q(N_q-2)-\boldsymbol{d}_q(N_q-2)]&\\ \le \sum_{i\in[\tilde{n}]} \mathbb{1}_{[(N_q-1)\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i)-s_q\mathbb{1}_{[(N_q-1)\in L_q]}&\\ \le \epsilon^{2le}+2[\boldsymbol{c}_q(N_q-1)-\boldsymbol{d}_q(N_q-1)]-[\boldsymbol{c}_q(N_q-2)-\boldsymbol{d}_q(N_q-2)]. \quad (15)\end{aligned}$$

We can eliminate the pair of carry $[\boldsymbol{c}_q(N_q-1)-\boldsymbol{d}_q(N_q-1)]$ by computing $2\times$ Eq. (14) + Eq. (15), and obtain

$$\begin{aligned}-(2^0+2^1)\epsilon^{2le}-[\boldsymbol{c}_q(N_q-2)-\boldsymbol{d}_q(N_q-2)]&\\ \le \sum_{l=\{N_q-1,N_q\}}\left(\sum_{i\in[\tilde{n}]}\mathbb{1}_{[l\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i)-s_q\mathbb{1}_{[l\in L_q]}\right)2^{l-(N_q-1)}&\\ \le (2^0+2^1)\epsilon^{2le}-[\boldsymbol{c}_q(N_q-2)-\boldsymbol{d}_q(N_q-2)]. \quad (16)\end{aligned}$$

By repeating the process until the equation of the lowest bit, we can eliminate all pairs of carry variables and obtain

$$-(2^0+\cdots+2^{N_q})\epsilon^{2le} \le \underbrace{\sum_{l=0}^{N_q}\left(\sum_{i\in[\tilde{n}]}\mathbb{1}_{[l\in L_q^i]}\cdot s_q^i\tilde{\boldsymbol{x}}(i)-s_q\mathbb{1}_{[l\in L_q]}\right)2^l}_{=\tilde{\boldsymbol{a}}_q^\top\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{b}}(q) \text{ by Eq. (7)}} \le (2^0+\cdots+2^{N_q})\epsilon^{2le}. \quad (17)$$

Hence, we can bound the qth linear equation in LEN by

$$\left|\tilde{\boldsymbol{a}}_q^\top\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{b}}(q)\right| \le (2^0+\cdots+2^{N_q})\epsilon^{2le} \le 2^{N_q+1}\epsilon^{2le},$$

which implies that the error of the qth equation of LENA is accumulated as a weighted sum of at most N_q equations in 2-LENA, where the weight is in the form of power of 2.

To bound all the linear equations in LENA uniformly, we have

$$\begin{aligned}\tau^{le} &= \max_{q\in[\tilde{m}]}\left|\tilde{\boldsymbol{a}}_q^\top\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{b}}(q)\right| &&\\ &\le \max_{q\in[\tilde{m}]} 2^{N_q+1}\epsilon^{2le} &&\\ &\le 2^N\epsilon^{2le} && \text{Because } N=1+\max_{q\in\tilde{m}}N_q \text{ as in Eq. (11)}\\ &\le 2^{1+\log X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})}\epsilon^{2le} && \text{Because } N=1+\left\lfloor\log X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\right\rfloor \text{ as in Eq. (11)}\\ &\le 2X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\epsilon^{2le}. &&\end{aligned}$$

As we set in the reduction that $\epsilon^{2le}=\frac{\epsilon^{le}}{2X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})}$, then we have

$$\tau^{le} \le 2X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})\frac{\epsilon^{le}}{2X(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{b}})}=\epsilon^{le},$$

which indicates that $\tilde{\boldsymbol{x}}$ is a solution to the LENA instance. □

4.3 2-LEN(A) to 1-LEN(A)

4.3.1 2-LEN to 1-LEN

We show the reduction from a 2-LEN instance $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{R}, 2)$ to a 1-LEN instance $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}, \hat{R}, 1)$. The 2-LEN instance has the form of $\bar{\boldsymbol{A}}\bar{\boldsymbol{x}} = \bar{\boldsymbol{b}}$, where entries of $\bar{\boldsymbol{A}}$ are integers between $[-2, 2]$. To reduce it to a 1-LEN instance, for each variable $\bar{\boldsymbol{x}}(j)$ that has a ± 2 coefficient, we introduce a new variable $\bar{\boldsymbol{x}}'(j)$, replace every $\pm 2\bar{\boldsymbol{x}}(j)$ with $\pm(\bar{\boldsymbol{x}}(j) + \bar{\boldsymbol{x}}'(j))$, and add an additional equation $\bar{\boldsymbol{x}}(j) - \bar{\boldsymbol{x}}'(j) = 0$.

If a 1-LEN solver returns $\hat{\boldsymbol{x}} = (\bar{\boldsymbol{x}}^\top, (\bar{\boldsymbol{x}}')^\top)^\top$ for the 1-LEN instance $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}, \hat{R}, 1)$, then we return $\bar{\boldsymbol{x}}$ for the 2-LEN instance $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{R}, 2)$; if the 1-LEN returns "infeasible" for the 1-LEN instance, then we return "infeasible" for the 2-LEN instance.

Lemma 4.7 (2-LEN to 1-LEN)**.** *Given a* 2-LEN *instance* $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{R}, 2)$ *where* $\bar{\boldsymbol{A}} \in \mathbb{Z}^{\bar{m} \times \bar{n}}, \bar{\boldsymbol{b}} \in \mathbb{Z}^{\bar{m}}$*, we can construct, in* $O(\mathrm{nnz}(\bar{\boldsymbol{A}}))$ *time, a* 1-LEN *instance* $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}, \hat{R}, 1)$ *where* $\hat{\boldsymbol{A}} \in \mathbb{Z}^{\hat{m} \times \hat{n}}, \hat{\boldsymbol{b}} \in \mathbb{Z}^{\hat{m}}$ *such that*

$$\hat{n} \le 2\bar{n}, \quad \hat{m} \le \bar{m} + \bar{n}, \quad \mathrm{nnz}(\hat{\boldsymbol{A}}) \le 4\,\mathrm{nnz}(\bar{\boldsymbol{A}}), \quad \hat{R} = 2\bar{R}, \quad X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}) = X(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}),$$

and if the 2-LEN *instance has a solution, then the* 1-LEN *instance has a solution.*

Proof. Based on the reduction described above, from any solution $\bar{\boldsymbol{x}}$ to the 2-LEN instance such that $\bar{\boldsymbol{A}}\bar{\boldsymbol{x}} = \bar{\boldsymbol{b}}$, we can derive a solution $\hat{\boldsymbol{x}} = (\bar{\boldsymbol{x}}^\top, (\bar{\boldsymbol{x}}')^\top)^\top$ to the 1-LEN instance. Concretely, for each $\bar{\boldsymbol{x}}(j)$ having a ± 2 coefficient in $\bar{\boldsymbol{A}}$, we set $\bar{\boldsymbol{x}}'(j) = \bar{\boldsymbol{x}}(j)$, where $\bar{\boldsymbol{x}}'(j)$ is the entry that we use to replace $\pm 2\bar{\boldsymbol{x}}(j)$ with $\pm(\bar{\boldsymbol{x}}_j + \bar{\boldsymbol{x}}'_j)$ in the reduction. We can check that $\hat{\boldsymbol{x}}$ is a solution to the 1-LEN instance.

Now, we track the change of problem size after reduction. Based on the reduction method, given a 2-LEN instance with $\bar{n}$ variables, $\bar{m}$ linear equations, and $\mathrm{nnz}(\bar{\boldsymbol{A}})$ nonzero entries, we can compute the size of the reduced 1-LEN instance as follows.

1. $\hat{n}$ variables, where
$$\hat{n} \le 2\bar{n}.$$
It is because each variable $\bar{\boldsymbol{x}}(j)$ in 2-LEN is replaced by at most 2 variables $\bar{\boldsymbol{x}}(j) + \bar{\boldsymbol{x}}'(j)$ in 1-LEN.

2. $\hat{m}$ linear constraints. In addition to the original $\bar{m}$ linear equations, each variable $\bar{\boldsymbol{x}}(j)$ with ± 2 coefficient in 2-LEN will introduce a new equation $\bar{\boldsymbol{x}}(j) - \bar{\boldsymbol{x}}'(j) = 0$ in 1-LEN. Thus,
$$\hat{m} \le \bar{m} + \bar{n}.$$

3. $\mathrm{nnz}(\hat{\boldsymbol{A}})$ nonzeros. To bound it, first, each nonzero entry in $\bar{\boldsymbol{A}}$ becomes at most two nonzero entries in $\hat{\boldsymbol{A}}$ because of the replacement of $\pm 2\bar{\boldsymbol{x}}(j)$ by $\pm(\bar{\boldsymbol{x}}(j) + \bar{\boldsymbol{x}}'(j))$. Next, at most $2\bar{n}$ new nonzero entries are generated because of the newly added equation $\bar{\boldsymbol{x}}(j) - \bar{\boldsymbol{x}}'(j) = 0$. Thus,
$$\mathrm{nnz}(\hat{\boldsymbol{A}}) \le 2(\mathrm{nnz}(\bar{\boldsymbol{A}}) + \bar{n}) \le 4\,\mathrm{nnz}(\bar{\boldsymbol{A}}),$$
where we use $\bar{n} \le \mathrm{nnz}(\bar{\boldsymbol{A}})$.

4. $\hat{R}$ radius of polytope in ℓ_1 norm. We have
$$\|\hat{\boldsymbol{x}}\|_1 = \|\bar{\boldsymbol{x}}\|_1 + \left\|\bar{\boldsymbol{x}}'\right\|_1 \le 2\,\|\bar{\boldsymbol{x}}\|_1 \le 2\bar{R}.$$
Hence, it suffices to set
$$\hat{R} = 2\bar{R}.$$

5. $X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}) = X(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}})$ because by construction,

$$\left\|\hat{\boldsymbol{A}}\right\|_{max} \leq \left\|\bar{\boldsymbol{A}}\right\|_{max}, \qquad \left\|\hat{\boldsymbol{b}}\right\|_{max} = \left\|\bar{\boldsymbol{b}}\right\|_{max}.$$

To estimate the reduction time, it takes constant time to deal with each $\bar{\boldsymbol{A}}(i,j)$ being ± 2, and there are at most $\mathrm{nnz}(\bar{\boldsymbol{A}})$ occurrences of ± 2 to be dealt with. Hence, it takes $O(\mathrm{nnz}(\bar{\boldsymbol{A}}))$ time to eliminate all the occurrence of ± 2. Moreover, copy the rest coefficients also takes $O(\mathrm{nnz}(\bar{\boldsymbol{A}}))$ time. Thus, the reduction of this step takes $O(\mathrm{nnz}(\bar{\boldsymbol{A}}))$ time.

□

4.3.2 2-LENA to 1-LENA

The above lemma shows the reduction between exactly solving a 2-LEN instance and exactly solving a 1-LEN instance. Next, we generalize the case with exact solutions to the case that allows approximate solutions.

We can use the same reduction method in the exact case to reduce a 2-LENA instance to a 1-LENA instance. We can also use the same solution mapping method, but we make a slight adjustment in the approximate case for the simplicity of the following error analysis. More specifically, if a 1-LENA solver returns $\hat{\boldsymbol{x}} = (\boldsymbol{x}^\top, (\boldsymbol{x}')^\top)^\top$ for the 1-LENA instance $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}, \hat{R}, 1, \epsilon^{1le})$, instead of returning $\boldsymbol{x}$ directly as a solution to the 2-LENA instance $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{R}, 2, \epsilon^{2le})$, we set $\bar{\boldsymbol{x}}(i) = \frac{1}{2}(\boldsymbol{x}(i) + \boldsymbol{x}'(i))$ if $\boldsymbol{x}(i)$ has a coefficient ± 2 in the 2-LEN instance, and set $\bar{\boldsymbol{x}}(i) = \boldsymbol{x}(i)$ otherwise. In addition, if the 1-LENA solver returns "infeasible" for the 1-LENA instance, then we return "infeasible" for the 2-LENA instance.

Lemma 4.8 (2-LENA to 1-LENA)**.** *Given a* 2-LENA *instance* $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{R}, 2, \epsilon^{2le})$ *where* $\bar{\boldsymbol{A}} \in \mathbb{Z}^{\bar{m}\times\bar{n}}, \bar{\boldsymbol{b}} \in \mathbb{Z}^{\bar{m}}$*, we can construct, in* $O(\mathrm{nnz}(\bar{\boldsymbol{A}}))$ *time, a* 1-LENA *instance* $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}, \hat{R}, 1, \epsilon^{1le})$ *where* $\hat{\boldsymbol{A}} \in \mathbb{Z}^{\hat{m}\times\hat{n}}, \hat{\boldsymbol{b}} \in \mathbb{Z}^{\hat{m}}$ *such that*

$$\hat{n} \leq 2\bar{n}, \quad \hat{m} \leq \bar{m} + \bar{n}, \quad \mathrm{nnz}(\hat{\boldsymbol{A}}) \leq 4\,\mathrm{nnz}(\bar{\boldsymbol{A}}), \quad \hat{R} = 2\bar{R}, \quad X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}) = X(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}),$$

$$\epsilon^{1le} = \frac{\epsilon^{2le}}{\bar{n}+1}.$$

If the 2-LEN *instance* $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}, \bar{R}, 2)$ *has a solution, then the* 1-LEN *instance* $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}, \hat{R}, 1)$ *has a solution. Furthermore, if* $\hat{\boldsymbol{x}}$ *is a solution to the* 1-LENA *instance, then in time* $O(\bar{n})$*, we can compute a solution* $\bar{\boldsymbol{x}}$ *to the* 2-LENA *instance.*

Proof. Since we use the same reduction method in the exact case to the approximate case, the conclusions in Lemma 4.7 also apply here, including the reduction time, problem size, and that 1-LEN has a feasible solution when 2-LEN has one. It remains to show the solution mapping time, as well as how the problem error changes by mapping an approximate solution to 1-LENA back to an approximate solution to 2-LENA.

Based on the solution mapping method described above, it takes constant time to set the value of each entry of $\bar{\boldsymbol{x}}$ by computing an averaging or mapping back trivially, and the size of $\bar{\boldsymbol{x}}$ is $\bar{n}$, thus the solution mapping takes $O(\bar{n})$ time.

Now, we conduct an error analysis. By Definition 4.5, the error of each linear equation in 1-LENA can be bounded by ϵ^{1le}. For a single occurrence of $\bar{\boldsymbol{A}}(i,j) = \pm 2$, we first bound the error of the equation $\bar{\boldsymbol{x}}(j) - \bar{\boldsymbol{x}}'(j) = 0$, and obtain

$$|\bar{\boldsymbol{x}}(j) - \bar{\boldsymbol{x}}'(j)| \leq \epsilon^{1le},$$

hence, we have

$$-\epsilon^{1le} \leq \bar{\boldsymbol{x}}(j) - \bar{\boldsymbol{x}}'(j) \leq \epsilon^{1le},$$

and thus

$$-\epsilon^{1le} + 2\bar{\boldsymbol{x}}(j) \leq \bar{\boldsymbol{x}}(j) + \bar{\boldsymbol{x}}'(j) \leq \epsilon^{1le} + 2\bar{\boldsymbol{x}}(j). \tag{18}$$

We first consider the case that, in the equation $\bar{\boldsymbol{a}}_i^\top \bar{\boldsymbol{x}} = \bar{\boldsymbol{b}}(i)$, there is only one entry j such that $\bar{\boldsymbol{a}}_i(j) = \pm 2$. By separating this term, we can write

$$\bar{\boldsymbol{a}}_i^\top \bar{\boldsymbol{x}} = \sum_{k \neq j} \bar{\boldsymbol{A}}(i,k)\bar{\boldsymbol{x}}(k) \pm 2\bar{\boldsymbol{x}}(j), \qquad \hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} = \sum_{k \neq j} \bar{\boldsymbol{A}}(i,k)\bar{\boldsymbol{x}}(k) \pm (\bar{\boldsymbol{x}}(j) + \bar{\boldsymbol{x}}'(j)).$$

Adding $\sum_{k\neq j} \bar{\boldsymbol{A}}(i,k)\bar{\boldsymbol{x}}(k)$ to Eq, (18), we have

$$\underbrace{-\epsilon^{1le} \pm 2\bar{\boldsymbol{x}}(j) + \sum_{k\neq j} \bar{\boldsymbol{A}}(i,k)\bar{\boldsymbol{x}}(k)}_{\bar{\boldsymbol{a}}_i^\top \bar{\boldsymbol{x}}} \leq \underbrace{\sum_{k\neq j} \bar{\boldsymbol{A}}(i,k)\bar{\boldsymbol{x}}(k) \pm (\bar{\boldsymbol{x}}(j) + \bar{\boldsymbol{x}}'(j))}_{\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}}} \leq \epsilon^{1le} \underbrace{\pm 2\bar{\boldsymbol{x}}(j) + \sum_{k\neq j} \bar{\boldsymbol{A}}(i,k)\bar{\boldsymbol{x}}(k)}_{\bar{\boldsymbol{a}}_i^\top \bar{\boldsymbol{x}}}.$$

And since $\hat{\boldsymbol{b}}(i) = \bar{\boldsymbol{b}}(i)$, we can subtract it from all parts and get

$$-\epsilon^{1le} + \bar{\boldsymbol{a}}_i^\top \bar{\boldsymbol{x}} - \bar{\boldsymbol{b}}(i) \leq \hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} - \hat{\boldsymbol{b}}(i) \leq \epsilon^{1le} + \bar{\boldsymbol{a}}_i^\top \bar{\boldsymbol{x}} - \bar{\boldsymbol{b}}(i).$$

which can be further transformed to

$$-\epsilon^{1le} + \hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} - \hat{\boldsymbol{b}}(i) \leq \bar{\boldsymbol{a}}_i^\top \bar{\boldsymbol{x}} - \bar{\boldsymbol{b}}(i) \leq \hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} - \hat{\boldsymbol{b}}(i) + \epsilon^{1le}. \tag{19}$$

If there are k_i occurrence of $\bar{\boldsymbol{A}}(i,j) = \pm 2$, then we can generalize Eq. (19) to

$$-k_i\epsilon^{1le} + \hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} - \hat{\boldsymbol{b}}(i) \leq \bar{\boldsymbol{a}}_i^\top \bar{\boldsymbol{x}} - \bar{\boldsymbol{b}}(i) \leq \hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} - \hat{\boldsymbol{b}}(i) + k_i\epsilon^{1le}, \tag{20}$$

hence, we can bound

$$\begin{aligned} \left|\bar{\boldsymbol{a}}_i^\top \bar{\boldsymbol{x}} - \bar{\boldsymbol{b}}(i)\right| &\leq k_i\epsilon^{1le} + \left|\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} - \hat{\boldsymbol{b}}(i)\right| \\ &\leq (k_i+1)\epsilon^{1le}, \end{aligned} \tag{21}$$

where the last inequality is because $\left|\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} - \hat{\boldsymbol{b}}(i)\right| \leq \epsilon^{1le}$ by applying the error of the 1-LENA instance.

To bound all the linear equations in 2-LENA uniformly, we have

$$\begin{aligned} \tau^{2le} &= \max_{i\in[\bar{m}]} \left|\bar{\boldsymbol{a}}_i^\top \bar{\boldsymbol{x}} - \bar{\boldsymbol{b}}(i)\right| \\ &\leq \max_{i\in[\bar{m}]} (k_i+1)\epsilon^{1le} && \text{Because of Eq. (21)} \\ &\leq (\bar{n}+1)\epsilon^{1le} && \text{Because } k_i \leq \bar{n} \end{aligned}$$

As we set in the reduction that $\epsilon^{1le} = \frac{\epsilon^{2le}}{\bar{n}+1}$, then we have

$$\tau^{2le} \leq (\bar{n}+1)\frac{\epsilon^{2le}}{\bar{n}+1} = \epsilon^{2le},$$

which indicates that $\bar{\boldsymbol{x}}$ is a solution to the 2-LENA instance.

□

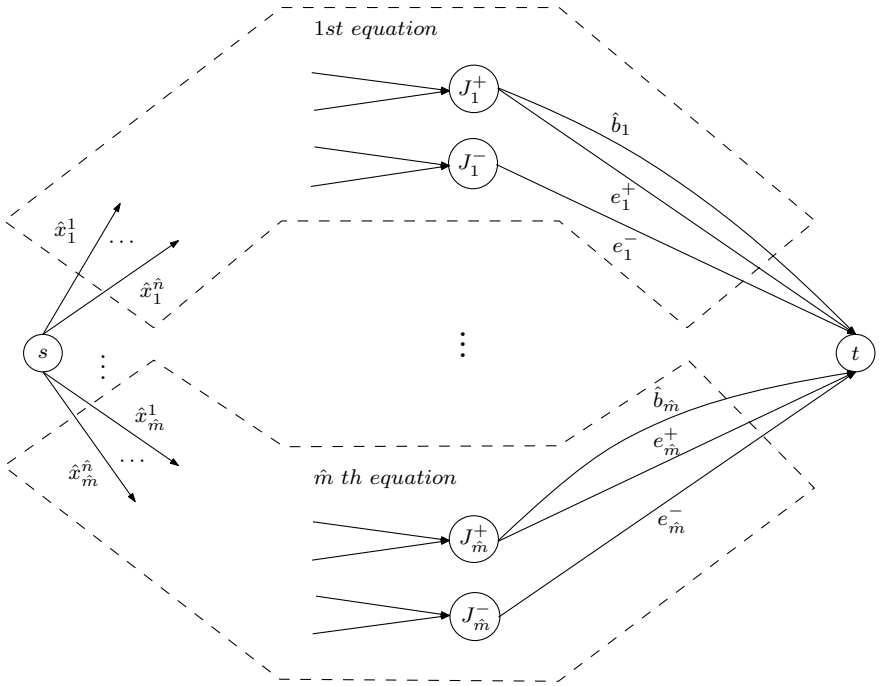

Figure 1: The reduction from 1-LEN to FHF.

4.4 1-LEN(A) to FHF(A)

4.4.1 1-LEN to FHF

The following is the approach to reduce a 1-LEN instance $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}, \hat{R}, 1)$ to an FHF instance $(G^h, F^h, \boldsymbol{u}^h, \mathcal{H}^h, s, t)$. The 1-LEN has the form of $\hat{\boldsymbol{A}}\hat{\boldsymbol{x}} = \hat{\boldsymbol{b}}$, where $\hat{\boldsymbol{A}} \in \mathbb{Z}^{\hat{m}\times\hat{n}}, \hat{\boldsymbol{b}} \in \mathbb{Z}^{\hat{m}}$. For an arbitrary equation i in the 1-LEN instance, $\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} = \hat{\boldsymbol{b}}(i), i \in [\hat{m}]$, let $J_i^+ = \{j|\hat{\boldsymbol{a}}_i(j) = 1\}$ and $J_i^- = \{j|\hat{\boldsymbol{a}}_i(j) = -1\}$ denote the set of indices of variables with coefficients being 1 and -1 in equation i, respectively. Then, each equation can be rewritten as a difference of the sum of variables with coefficient 1 and -1:

$$\sum_{j\in J_i^+} \hat{\boldsymbol{x}}(j) - \sum_{j\in J_i^-} \hat{\boldsymbol{x}}(j) = \hat{\boldsymbol{b}}(i), \quad i \in [\hat{m}]. \tag{22}$$

We claim that $\hat{\boldsymbol{A}}\hat{\boldsymbol{x}} = \hat{\boldsymbol{b}}$ can be represented by a graph that is composed of a number of homologous edges and fixed flow edges, as shown in Figure 1.

More specifically, the fixed homologous flow network consists of a source s, a sink t, and $\hat{m}$ sections such that each section i represents the ith linear equation in 1-LEN, as shown in Eq. (22).

Inside each section i, there are 2 vertices $\{J_i^-, J_i^+\}$ and a number of edges:

- For the incoming edges of $\{J_i^-, J_i^+\}$,
  - if $\hat{\boldsymbol{a}}_i(j) = 1$, then s is connected to J_i^+ by edge $\hat{x}_i^j$ with capacity $\hat{R}$;
  - if $\hat{\boldsymbol{a}}_i(j) = -1$, then s is connected to J_i^- by edge $\hat{x}_i^j$ with capacity $\hat{R}$;
  - if $\hat{\boldsymbol{a}}_i(j) = 0$, no edge is needed.

  Note that the problem sparsity is preserved in the graph construction. The amount of flow routed in these incoming edges equals the value of the corresponding variables. To ensure the

consistency of the value of variables over $\hat{m}$ equations, those incoming edges that correspond to the same variable are forced to route the same amount of flow, i.e., $(\hat{x}_1^j, \cdots, \hat{x}_{\hat{m}}^j), j \in [\hat{n}]$ constitute a homologous edge set that corresponds to the variable $\hat{\boldsymbol{x}}(j)$. Note that the size of such a homologous edge set is at most $\hat{m}$.

- For the outgoing edges of $\{J_i^-, J_i^+\}$,
  - J_i^+ is connected to t by a fixed flow edge $\hat{b}_i$ that routes $\hat{\boldsymbol{b}}(i)$ units of flow;
  - J_i^+ and J_i^- are connected to t by a pair of homologous edges e_i^+, e_i^- with capacity $\hat{R}$.

If an FHF solver returns $\boldsymbol{f}^h$ for the FHF instance $(G^h, F^h, \boldsymbol{u}^h, \mathcal{H}^h, s, t)$, then we return $\hat{\boldsymbol{x}}$ for the 1-LEN instance $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}, \hat{R}, 1)$, by setting for every $j \in [\hat{n}]$,

$$\hat{\boldsymbol{x}}(j) = \boldsymbol{f}^h(\hat{x}_i^j), \text{ for an arbitrary } i \in [\hat{m}]$$

If the FHF solver returns "infeasible" for the FHF instance, then we return "infeasible" for the 1-LEN instance.

Lemma 4.9 (1-LEN to FHF)**.** *Given a* 1-LEN *instance* $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}, \hat{R}, 1)$ *where* $\hat{\boldsymbol{A}} \in \mathbb{Z}^{\hat{m}\times\hat{n}}, \hat{\boldsymbol{b}} \in \mathbb{Z}^{\hat{m}}$*, we we can construct, in time* $O(\mathrm{nnz}(\hat{\boldsymbol{A}}))$*, an* FHF *instance* $(G^h, F^h, \boldsymbol{u}^h, \mathcal{H}^h = (H_1, \cdots, H_h), s, t)$ *such that*

$$|V^h| = 2\hat{m} + 2, \quad |E^h| \le 4\,\mathrm{nnz}(\hat{\boldsymbol{A}}), \quad |F^h| = \hat{m}, \quad h = \hat{n} + \hat{m}, \quad \left\|\boldsymbol{u}^h\right\|_{max} = \max\left\{\hat{R}, X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}})\right\},$$

and if the 1-LEN *instance has a solution, then the* FHF *instance has a solution.*

Proof. According to the reduction described above, from any solution $\hat{\boldsymbol{x}}$ to the 1-LEN instance such that $\hat{\boldsymbol{A}}\hat{\boldsymbol{x}} = \hat{\boldsymbol{b}}$, we can derive a solution $\boldsymbol{f}^h$ to the FHF instance. Concretely, we define a feasible flow $\boldsymbol{f}^h$ as follows:

- For incoming edges of $\{J_i^-, J_i^+\}$, we set

  $$\boldsymbol{f}^h(\hat{x}_1^j) = \cdots = \boldsymbol{f}^h(\hat{x}_{\hat{m}}^j) = \hat{\boldsymbol{x}}(j) \le \hat{R}, \qquad \forall j \in [\hat{n}],$$

  which satisfies the homologous constraint for the homologous edge sets $(\hat{x}_1^j, \cdots, \hat{x}_{\hat{m}}^j), j \in [\hat{n}]$, as well as the capacity constraint.

- For outgoing edges of $\{J_i^-, J_i^+\}$, we set

  $$\boldsymbol{f}^h(\hat{b}_i) = \hat{\boldsymbol{b}}(i), \qquad \forall i \in \hat{m},$$

  which satisfies the fixed flow constraint for edges $\hat{b}_1, \cdots, \hat{b}_{\hat{m}}$; and set

  $$\boldsymbol{f}^h(e_i^+) = \boldsymbol{f}^h(e_i^-) = \sum_{j\in J_i^-} \hat{\boldsymbol{x}}(j) = \sum_{j\in J_i^+} \hat{\boldsymbol{x}}(j) - \hat{\boldsymbol{b}}(j),$$

  which satisfies the homologous constraint for edge e_i^+, e_i^-, and the conservation of flows for vertices J_i^+, J_i^-.

Therefore, we conclude that $\boldsymbol{f}^h$ is a feasible flow to the FHF instance.

Now, we track the change of problem size after reduction. Based on the reduction method, given a 1-LENA instance with $\hat{n}$ variables, $\hat{m}$ linear equations, and $\mathrm{nnz}(\hat{\boldsymbol{A}})$ nonzero entries, it is straightforward to get the size of the reduced FHFA instance as follows.

1. $|V^h|$ vertices, where
$$|V^h| = 2\hat{m} + 2.$$

2. $|E^h|$ edges, where
$$|E^h| = \operatorname{nnz}(\hat{\boldsymbol{A}}) + 3\hat{m} \le 4\operatorname{nnz}(\hat{\boldsymbol{A}}),$$
since $\hat{m} \le \operatorname{nnz}(\hat{\boldsymbol{A}})$.

3. $|F^h|$ fixed flow edges, where
$$|F^h| = \hat{m}.$$

4. h homologous edge sets, where
$$h = \hat{m} + \hat{n}.$$

5. The maximum edge capacity is bounded by
$$\left\|\boldsymbol{u}^h\right\|_{max} = \max\left\{\hat{R}, \left\|\hat{\boldsymbol{b}}\right\|_{max}\right\} \le \max\left\{\hat{R}, X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}})\right\}.$$

To estimate the reduction time, as there are $|V^h| = O(\hat{m})$ vertices and $|E^h| = O(\operatorname{nnz}(\hat{\boldsymbol{A}}))$ edges in G^h, thus, performing such a reduction takes $O(\operatorname{nnz}(\hat{\boldsymbol{A}}))$ time to construct G^h. □

4.4.2 1-LENA to FHFA

The above lemma shows the reduction between exactly solving a 1-LEN instance and exactly solving a FHF instance. Next, we generalize the case with exact solutions to the case that allows approximate solutions. First of all, we give a definition of the approximate version of FHF.

Definition 4.10 (FHF Approximate Problem (FHFA))**.** An FHFA instance is given by an FHF instance $(G, F, \boldsymbol{u}, \mathcal{H}, s, t)$ as in Definition 2.12, and error parameters $\epsilon_l, \epsilon_u, \epsilon_d, \epsilon_h \in [0, 1]$, which we collect in a tuple $(G, F, \mathcal{H}, \boldsymbol{u}, s, t, \epsilon_l, \epsilon_u, \epsilon_d, \epsilon_h)$. We say an algorithm solves the FHFA problem, if, given any FHFA instance, it returns a flow $\boldsymbol{f} \ge \boldsymbol{0}$ that satisfies

$$(1-\epsilon_l)\boldsymbol{u}(e) \le \boldsymbol{f}(e) \le (1+\epsilon_u)\boldsymbol{u}(e),\ \forall e \in F \tag{23}$$
$$0 \le \boldsymbol{f}(e) \le (1+\epsilon_u)\boldsymbol{u}(e),\ \forall e \in E\backslash F \tag{24}$$
$$\left|\sum_{u:(u,v)\in E} \boldsymbol{f}(u,v) - \sum_{w:(v,w)\in E} \boldsymbol{f}(v,w)\right| \le \epsilon_d,\ \forall v \in V\backslash\{s,t\} \tag{25}$$
$$\left|\boldsymbol{f}(v,w) - \boldsymbol{f}(v',w')\right| \le (k-1)\epsilon_h,\ \forall (v,w),(v',w') \in H_i,\ |H_i| = k \tag{26}$$

or it correctly declares that the associated FHF instance is infeasible. We refer to the error in (23) and (24) as error in congestion, error in (25) as error in demand, and error in (26) as error in homology.

Remark. Note that error in demand in FHFA is measured additively, which is defined as the net flow of all the vertices other than $\{s, t\}$. This error is caused by mapping a solution to FPHFA to a solution to FHFA. Also note that error in demand for $\{s, t\}$ are not defined since it is a single commodity problem and no requirements are imposed on the flow value.

We can use the same reduction method and solution mapping method in the exact case to reduce a 1-LENA instance to an FHFA instance. Note that, though we still obtain $\hat{\boldsymbol{x}}$ by setting for each $j \in [\hat{n}]$,
$$\hat{\boldsymbol{x}}(j) = \boldsymbol{f}^h(\hat{x}^i_j), \qquad \text{for an arbitrary } i \in [\hat{m}]$$

Lemma 4.11 (1-LENA to FHFA)**.** *Given a* 1-LENA *instance* $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}, \hat{R}, 1, \epsilon^{1le},)$ *where* $\hat{\boldsymbol{A}} \in \mathbb{Z}^{\hat{m}\times\hat{n}}, \hat{\boldsymbol{b}} \in \mathbb{Z}^{\hat{m}}$*, we can construct, in* $O(\mathrm{nnz}(\hat{\boldsymbol{A}}))$ *time, an* FHFA *instance* $(G^h, F^h, \mathcal{H}^h = (H_1, \cdots, H_h), \boldsymbol{u}^h, s, t, \epsilon_l^h, \epsilon_u^h, \epsilon_d^h, \epsilon_h^h)$ *such that*

$$|V^h| = 2\hat{m} + 2, \quad |E^h| \le 4\,\mathrm{nnz}(\hat{\boldsymbol{A}}), \quad |F^h| = \hat{m}, \quad h = \hat{n} + \hat{m}, \quad \left\|\boldsymbol{u}^h\right\|_{max} = \max\left\{\hat{R}, X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}})\right\},$$

and

$$\epsilon_l^h = \frac{\epsilon^{1le}}{3X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}})}, \quad \epsilon_u^h = \frac{\epsilon^{1le}}{3X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}})}, \quad \epsilon_d^h = \frac{\epsilon^{1le}}{6}, \quad \epsilon_h^h = \frac{\epsilon^{1le}}{3\hat{n}\hat{m}X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}})}.$$

If the 1-LEN *instance* $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}, \hat{R}, 1)$ *has a solution, then the* FHF *instance* $(G^h, F^h, \mathcal{H}^h, \boldsymbol{u}^h, s, t)$ *has a solution. Furthermore, if* $\boldsymbol{f}^h$ *is a solution to the* FHFA *instance, then in time* $O(\hat{n})$*, we can compute a solution* $\hat{\boldsymbol{x}}$ *to the* 1-LENA *instance.*

Proof. Since we use the same reduction method in the exact case to the approximate case, the conclusions in Lemma 4.9 also apply here, including the reduction time, problem size, and that FHF has a feasible solution when 1-LEN has one. It remains to show the solution mapping time, as well as how the problem error changes by mapping an approximate solution to FHFA back to an approximate solution to 1-LENA.

Based on the solution mapping method described above, it takes constant time to set the value of each entry of $\hat{\boldsymbol{x}}$ as $\boldsymbol{f}^h$ on certain edges. As $\hat{\boldsymbol{x}}$ has $\hat{n}$ entries, such a solution mapping takes $O(\hat{n})$ time.

Now, we conduct an error analysis. We firstly investigate the error in the ith linear equation in the 1-LENA instance $\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} = \hat{\boldsymbol{b}}(i)$. Let $\hat{\boldsymbol{x}}_i(j) = \boldsymbol{f}^h(\hat{x}_i^j)$ denote the amount of flow routed through in the ith section of G^h. For each edge $\hat{x}_i^j$ in the ith section of G^h, by error in homology defined in Eq. (26), we have

$$|\hat{\boldsymbol{x}}(j) - \hat{\boldsymbol{x}}_i(j)| \le (\hat{m} - 1)\epsilon_h^h, \qquad \forall j \in [\hat{n}]$$

because the homologous edge set corresponding to the variable $\hat{\boldsymbol{x}}(j)$ has size at most $\hat{m}$. Thus, we can bound $|\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} - \hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}}_i|$ by

$$|\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} - \hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}}_i| = |\hat{\boldsymbol{a}}_i^\top (\hat{\boldsymbol{x}} - \hat{\boldsymbol{x}}_i)| \le (\hat{m} - 1)\epsilon_h^h \left\|\hat{\boldsymbol{a}}_i\right\|_1. \tag{27}$$

Next, we bound $\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}}_i$. We use $\boldsymbol{f}^h(s, J_i^\pm)$ to denote the total incoming flow of vertex $J_i^\pm$. Then, we have

- $\boldsymbol{f}^h(\hat{b}_i) \in \left[(1 - \epsilon_l^h)\hat{\boldsymbol{b}}(i), (1 + \epsilon_u^h)\hat{\boldsymbol{b}}(i)\right]$ because of error in congestion of fixed flow edges;
- $\left|\boldsymbol{f}^h(s, J_i^+) - (\boldsymbol{f}^h(e_i^+) + \boldsymbol{f}^h(\hat{b}_i))\right| \le \epsilon_d^h$ because of error in demand of vertex J_i^+;
- $\left|\boldsymbol{f}^h(s, J_i^-) - \boldsymbol{f}^h(e_i^-)\right| \le \epsilon_d^h$ because of error in demand of vertex J_i^-;
- $|\boldsymbol{f}^h(e_i^+) - \boldsymbol{f}^h(e_i^-)| \le (2 - 1)\epsilon_h^h = \epsilon_h^h$ because of error in homology between edge e_i^+, e_i^-.

Then,

$$\begin{aligned}
\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}}_i - \hat{\boldsymbol{b}}(i) &= \boldsymbol{f}^h(s, J_i^+) - \boldsymbol{f}^h(s, J_i^-) - \hat{\boldsymbol{b}}(i) \\
&\le 2\epsilon_d^h + (\boldsymbol{f}^h(e_i^+) - \boldsymbol{f}^h(e_i^-)) + (\boldsymbol{f}^h(\hat{b}_i) - \hat{\boldsymbol{b}}(i)) && \text{By flow conservation} \\
&\le 2\epsilon_d^h + \epsilon_h^h + \epsilon_u^h \hat{\boldsymbol{b}}(i),
\end{aligned}$$

and similarly,

$$\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}}_i - \hat{\boldsymbol{b}}(i) \geq -2\epsilon_d^h - \epsilon_h^h - \epsilon_l^h \hat{\boldsymbol{b}}(i).$$

Combining with Eq. (27), we have

$$\begin{aligned}
|\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} - \hat{\boldsymbol{b}}(i)| &= |\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} - \hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}}_i + \hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}}_i - \hat{\boldsymbol{b}}(i)| \\
&\leq |\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} - \hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}}_i| + |\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}}_i - \hat{\boldsymbol{b}}(i)| \\
&\leq (\hat{m}-1)\epsilon_h^h \left\|\hat{\boldsymbol{a}}_i\right\|_1 + \max\{\epsilon_l^h, \epsilon_u^h\}\hat{\boldsymbol{b}}(i) + 2\epsilon_d^h + \epsilon_h^h.
\end{aligned}$$

To bound all the linear equations in 1-LENA uniformly, we have

$$\begin{aligned}
\tau^{1le} &\stackrel{\text{def}}{=} \max_{i\in[\hat{m}]} |\hat{\boldsymbol{a}}_i^\top \hat{\boldsymbol{x}} - \hat{\boldsymbol{b}}(i)| \\
&\leq \max_{i\in[\hat{m}]} \left\{(\hat{m}-1)\epsilon_h^h \left\|\hat{\boldsymbol{a}}_i\right\|_1 + \max\{\epsilon_l^h, \epsilon_u^h\}\hat{\boldsymbol{b}}(i) + 2\epsilon_d^h + \epsilon_h^h\right\} \\
&\leq (\hat{m}-1)\epsilon_h^h \cdot \hat{n}X(\hat{\boldsymbol{A}}) + \max\{\epsilon_l^h, \epsilon_u^h\}X(\hat{\boldsymbol{b}}) + 2\epsilon_d^h + \epsilon_h^h \\
&\leq (\hat{m}-1)\epsilon_h^h \cdot \hat{n}X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}) + \max\{\epsilon_l^h, \epsilon_u^h\}X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}) + 2\epsilon_d^h + \epsilon_h^h \\
&\leq \epsilon_h^h \cdot \hat{n}\hat{m}X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}) + \max\{\epsilon_l^h, \epsilon_u^h\}X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}) + 2\epsilon_d^h. \qquad \text{Because } \hat{n}, X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}) \geq 1
\end{aligned}$$

As we set in the reduction that $\epsilon_l^h = \frac{\epsilon^{1le}}{3X(\hat{\boldsymbol{A}},\hat{\boldsymbol{b}})}$, $\epsilon_u^h = \frac{\epsilon^{1le}}{3X(\hat{\boldsymbol{A}},\hat{\boldsymbol{b}})}$, $\epsilon_d^h = \frac{\epsilon^{1le}}{6}$, $\epsilon_h^h = \frac{\epsilon^{1le}}{3\hat{n}\hat{m}X(\hat{\boldsymbol{A}},\hat{\boldsymbol{b}})}$, then we have

$$\begin{aligned}
\tau^{1le} &\leq \frac{\epsilon^{1le}}{3\hat{n}\hat{m}X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}})} \cdot \hat{n}\hat{m}X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}) + \frac{\epsilon^{1le}}{3X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}})} \cdot X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}) + 2 \cdot \frac{\epsilon^{1le}}{6} \\
&= \frac{\epsilon^{1le}}{3} + \frac{\epsilon^{1le}}{3} + \frac{\epsilon^{1le}}{3} = \epsilon^{1le},
\end{aligned}$$

which indicates that $\hat{\boldsymbol{x}}$ is a solution to the 1-LENA instance. □

4.5 FHF(A) to FPHF(A)

4.5.1 FHF to FPHF

We show the reduction from an FHF instance $(G^h, H^h, \mathcal{H}^h, \boldsymbol{u}^h, s, t)$ to an FPHF instance $(G^p, F^p, \mathcal{H}^p, \boldsymbol{u}^p, s, t)$. Suppose that $(v_1, w_1), \cdots, (v_k, w_k) \in E^h$ belong to a homologous edge set of size k in G^h. As shown in Figure 2[7], we replace $(v_i, w_i), i \in \{2, \cdots, k-1\}$ by two edges (v_i, z_i) and (z_i, w_i) such that z_i is a new vertex incident only to these two edges, and edge capacities of the two new edges are the same as that of the original edge (v_i, w_i). Then, we can construct $k-1$ pairs of homologous edges: (v_1, w_1) and (v_2, z_2); (z_2, w_2) and (v_3, z_3); $\cdots$; (z_i, w_i) and (v_{i+1}, z_{i+1}); $\cdots$; (z_{k-1}, w_{k-1}) and (v_k, w_k). In addition, no reduction is performed on non-homologous edges in G^h, and we trivially copy these edges to G^p.

If an FPHF solver returns $\boldsymbol{f}^p$ for the FPHF instance $(G^p, F^p, \mathcal{H}^p, \boldsymbol{u}^p, s, t)$, then we return $\boldsymbol{f}^h$ for the FHF instance $(G^h, H^h, \mathcal{H}^h, \boldsymbol{u}^h, s, t)$, by setting $\boldsymbol{f}^h(e) = \boldsymbol{f}^p(e_1)$ if e is a homologous edge and splits into two edges e_1, e_2 in the reduction such that e_1 precedes e_2, and setting $\boldsymbol{f}^h(e) = \boldsymbol{f}^p(e)$ otherwise. If the FPHF solver returns "infeasible" for the FPHF instance, then we return "infeasible" for the FHF instance.

[7] We use $(0, u)$ for an non-fixed flow edge of capacity u.

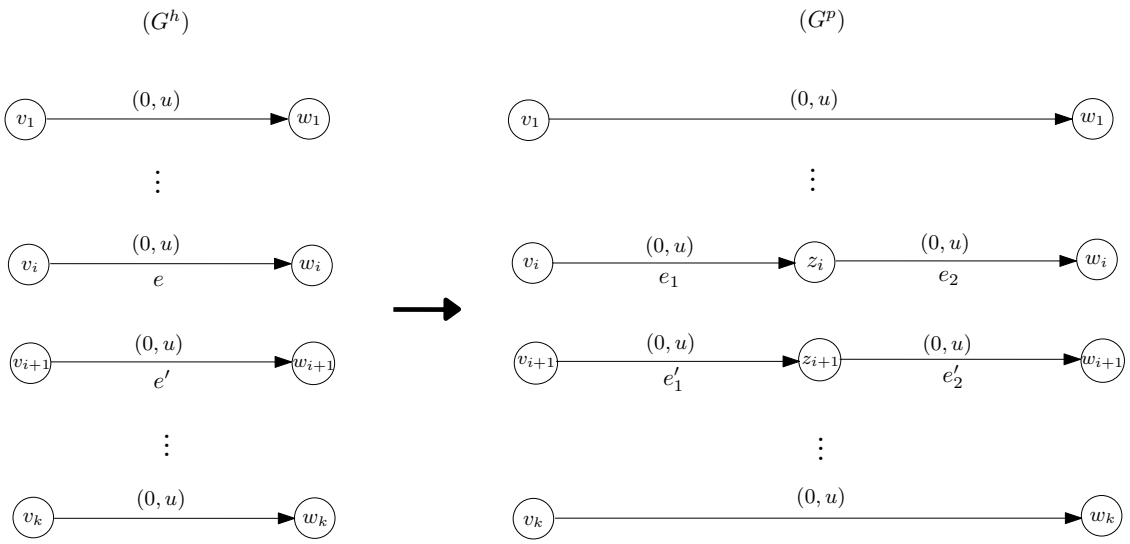

Figure 2: The reduction from FHF to FPHF.

Lemma 4.12 (FHF to FPHF)**.** *Given an* FHF *instance* $(G^h, F^h, \mathcal{H}^h = (H_1, \cdots, H_h), \boldsymbol{u}^h, s, t)$*, we can construct, in time* $O(|E^h|)$*, an* FPHF *instance* $(G^p, F^p, \mathcal{H}^p = (H_1, \cdots, H_p), \boldsymbol{u}^p, s, t)$ *such that*

$$|V^p| \le |V^h| + |E^h|, \qquad |E^p| \le 2|E^h|, \qquad |F^p| = |F^h|, \qquad p \le |E^h|, \qquad \|\boldsymbol{u}^p\|_{max} = \left\|\boldsymbol{u}^h\right\|_{max},$$

and if the FHF *instance has a solution, then the* FPHF *instance has a solution.*

Proof. According to the reduction described above, from any solution $\boldsymbol{f}^h$ to the FHF instance, it is easy to derive a solution $\boldsymbol{f}^p$ to the FPHF instance. Concretely, for any homologous edge $e \in E^h$ that is split into two edges $e_1, e_2 \in E^p$, we set

$$\boldsymbol{f}^p(e_1) = \boldsymbol{f}^p(e_2) = \boldsymbol{f}^h(e).$$

Since the vertex between e_1 and e_2 is only incident to these two edges, then the conservation of flows is satisfied on the inserted vertices. Moreover, since the edge capacity of e_1 and e_2 are the same as that of e, and they route the same amount of flows, thus the capacity constraint is also satisfied on the split edges. In addition, conservation of flows and capacity constraints are trivially satisfied for the rest vertices and edges. Therefore, $\boldsymbol{f}^p$ is a feasible flow to the FPHF instance.

Now, we track the change of problem size after reduction. Based on the reduction method, given an FHF instance with $|V^h|$ vertices, $|E^h|$ edges (including $|H^h|$ fixed flow edges, and h homologous edge sets $\mathcal{H}^h = \{H_1, \cdots, H_h\}$), we can compute the size of the reduced FPHFA instance as follows.

1. $|V^p|$ vertices. Since all vertices in V^h are maintained, and for each homologous edge set $H_i \in \mathcal{H}^h$, $|H_i| - 2$ new vertices are inserted, thus we have

$$\begin{aligned} |V^p| &= |V^h| + \sum_{i\in[h]} (|H_i| - 2) \\ &\le |V^h| + |E^h|. \qquad \text{Because } \sum_{i\in[h]} |H_i| \le |E^h| \end{aligned}$$

2. $|E^p|$ edges. Since all edges in E^h are maintained and a new edge is generated by inserting a new vertex, thus we have

$$|E^p| = |E^h| + \sum_{i\in[h]}(|H_i| - 2) \le 2|E^h|.$$

3. $|F^p|$ fixed flow edges, where

$$|F^p| = |F^h|.$$

It is because all fixed flow edges in F^h are maintained, and no new fixed flow edges are generated by reduction of this step.

4. p pairs of homologous edges. Since each homologous edge set $H_i \in \mathcal{H}^h$ can be transformed into $|H_i| - 1$ pairs, then we have

$$p = \sum_{i\in[h]}(|H_i| - 1) \le |E^h|.$$

5. The maximum edge capacity is bounded by

$$\|\boldsymbol{u}^p\|_{max} = \left\|\boldsymbol{u}^h\right\|_{max},$$

because the reduction of this step only separate edges by inserting new vertices without modifying edge capacities.

To estimate the reduction time, inserting a new vertex takes a constant time and there are $O(|E^h|)$ new vertices to be inserted. Also, it takes constant time to copy each of the rest $O(|E^h|)$ non-homologous edges. Hence, the reduction of this step takes $O(|E^h|)$ time. □

4.5.2 FHFA to FPHFA

The above lemma shows the reduction between exactly solving an FHF instance and exactly solving an FPHF instance. Next, we generalize the case with exact solutions to the case that allows approximate solutions. First of all, we give a definition of the approximate version of FPHF.

Definition 4.13 (FPHF Approximate Problem (FPHFA))**.** An FPHFA instance is given by an FPHF instance $(G, F, \mathcal{H} = \{H_1, \cdots, H_p\}, \boldsymbol{u}, s, t)$ as in Definition 2.13, and error parameters $\epsilon_l, \epsilon_u, \epsilon_d, \epsilon_h \in [0, 1]$, which we collect in a tuple $(G, F, \mathcal{H}, \boldsymbol{u}, s, t, \epsilon_l, \epsilon_u, \epsilon_d, \epsilon_h)$. We say an algorithm solves the FPHFA problem, if, given any FPHFA instance, it returns a flow $\boldsymbol{f} \ge \boldsymbol{0}$ that satisfies

$$(1 - \epsilon_l)\boldsymbol{u}(e) \le \boldsymbol{f}(e) \le (1 + \epsilon_u)\boldsymbol{u}(e),\ \forall e \in F \tag{28}$$

$$0 \le \boldsymbol{f}(e) \le (1 + \epsilon_u)\boldsymbol{u}(e),\ \forall e \in E\backslash F \tag{29}$$

$$\left|\sum_{u:(u,v)\in E} \boldsymbol{f}(u, v) - \sum_{w:(v,w)\in E} \boldsymbol{f}(v, w)\right| \le \epsilon_d,\ \forall v \in V\backslash\{s, t\} \tag{30}$$

$$|\boldsymbol{f}(v, w) - \boldsymbol{f}(y, z)| \le \epsilon_h,\ \forall H_i \ni (v, w), (y, z), i \in [p] \tag{31}$$

or it correctly declares that the associated FPHF instance is infeasible. We refer to the error in (28) and (29) as error in congestion, error in (30) as error in demand, and error in (31) as error in pair homology.

We can use the same reduction and solution mapping method in the exact case to the approximate case.

Lemma 4.14 (FHFA to FPHFA)**.** *Given an* FHFA *instance* $(G^h, F^h, \mathcal{H}^h = (H_1, \cdots, H_h), \boldsymbol{u}^h, s, t, \epsilon_l^h, \epsilon_u^h, \epsilon_d^h, \epsilon_h^h)$*, we can construct, in* $O(|E^h|)$ *time, an* FPHFA *instance* $(G^p, F^p, \mathcal{H}^p = (H_1, \cdots, H_p), \boldsymbol{u}^p, s, t, \epsilon_l^p, \epsilon_u^p, \epsilon_d^p, \epsilon_h^p)$ *such that*

$$|V^p| \leq |V^h| + |E^h|, \qquad |E^p| \leq 2|E^h|, \qquad |F^p| = |F^h|, \qquad p \leq |E^h|, \qquad \|\boldsymbol{u}^p\|_{max} = \left\|\boldsymbol{u}^h\right\|_{max},$$

and

$$\epsilon_l^p = \epsilon_l^h, \qquad \epsilon_u^p = \epsilon_u^h, \qquad \epsilon_d^p = \frac{\epsilon_d^h}{|E^h|}, \qquad \epsilon_h^p = \epsilon_h^h.$$

If the FHF *instance* $(G^h, H^h, \mathcal{H}^h, \boldsymbol{u}^h, s, t)$ *has a solution, then the* FPFH *instance* $(G^p, F^p, \mathcal{H}^p, \boldsymbol{u}^p, s, t)$ *has a solution. Furthermore, if* $\boldsymbol{f}^p$ *is a solution to the* FPHFA *instance, then in time* $O(|E^h|)$*, we can compute a solution* $\boldsymbol{f}^h$ *to the* FHFA *instance.*

Proof. Since we use the same reduction method in the exact case to the approximate case, the conclusions in Lemma 4.12 also apply here, including the reduction time, problem size, and that FPHF has a feasible solution when FHF has one. It remains to show the solution mapping time, as well as how the problem error changes by mapping an approximate solution to FPHFA back to an approximate solution to FHFA.

Based on the solution mapping method described above, it takes constant time to set the value of each homologous or non-homologous entry of $\boldsymbol{f}^h$, and $\boldsymbol{f}^h$ has $|E^h|$ entries, such a solution mapping takes $O(|E^h|)$ time.

Now, we conduct an error analysis. We claim that the mapping will generate three types of error: (1) error in congestion; (2) error in demand; (3) error in homology.

1. Error in congestion.
   We first track the error of the upper bound of capacity. For any homologous edge e, the maximum flow to be routed in e is

   $$(1 + \tau_u^h)\boldsymbol{u}^h(e) = \boldsymbol{f}^p(e_1) \leq (1 + \epsilon_u^p)\boldsymbol{u}^p(e_1) = (1 + \epsilon_u^p)\boldsymbol{u}^h(e),$$

   thus,

   $$\tau_u^h \leq \epsilon_u^p.$$

   As we set $\epsilon_u^p = \epsilon_u^h$ in the reduction, then we have

   $$\tau_u^h \leq \epsilon_u^h.$$

   This τ_u^h also applies to fixed flow edges since no reduction is made on them.

   Next, we track the error of the lower bound of capacity. Since only fixed flow edges have lower bound of capacity, and no reduction is made on fixed flow edges, then we obtain trivially that $\tau_l^h \leq \epsilon_l^p$. By setting $\epsilon_l^p = \epsilon_l^h$ in the reduction, we have

   $$\tau_l^h \leq \epsilon_l^h.$$

2. Error in demand.
   For simplicity, we denote $H^h = \bigcup_{i \in [h]} H_i$ as the set of all homologous edges. By Eq. (25) in

Definition 4.10, the error in demand that we can achieve for vertices other than s, t in G^h is computed as

$$
\begin{aligned}
\tau_d^h &= \max_{w\in V^h\setminus\{s,t\}} \left| \sum_{(v,w)\in E^h} \boldsymbol{f}^h(v,w) - \sum_{(w,u)\in E^h} \boldsymbol{f}^h(w,u) \right| \\
&\overset{(1)}{=} \max_{w\in V^h\setminus\{s,t\}} \left| \left(\sum_{(v,w)\in H^h} \boldsymbol{f}^h(v,w) + \sum_{(v,w)\in E^h\setminus H^h} \boldsymbol{f}^h(v,w) \right) - \left(\sum_{(w,u)\in H^h} \boldsymbol{f}^h(w,u) + \sum_{(w,u)\in E^h\setminus H^h} \boldsymbol{f}^h(w,u) \right) \right| \\
&\overset{(2)}{=} \max_{w\in V^h\setminus\{s,t\}} \left| \left(\sum_{e=(v,w)\in H^h} \boldsymbol{f}^p(e_1) + \sum_{(v,w)\in E^h\setminus H^h} \boldsymbol{f}^p(v,w) \right) - \left(\sum_{e=(w,u)\in H^h} \boldsymbol{f}^p(e_1) + \sum_{(w,u)\in E^h\setminus H^h} \boldsymbol{f}^p(w,u) \right) \right| \\
&\overset{(3)}{\leq} \epsilon_d^p + \max_{w\in V^h\setminus\{s,t\}} \left| \left(\sum_{e=(v,w)\in H^h} \boldsymbol{f}^p(e_1) + \sum_{(v,w)\in E^h\setminus H^h} \boldsymbol{f}^p(v,w) \right) - \left(\sum_{e=(v,w)\in H^h} \boldsymbol{f}^p(e_2) + \sum_{(v,w)\in E^h\setminus H^h} \boldsymbol{f}^p(v,w) \right) \right| \\
&= \epsilon_d^p + \max_{w\in V^h\setminus\{s,t\}} \left| \sum_{e=(v,w)\in H^h} \boldsymbol{f}^p(e_1) - \sum_{e=(v,w)\in H^h} \boldsymbol{f}^p(e_2) \right| \\
&= \epsilon_d^p + \max_{w\in V^h\setminus\{s,t\}} \left| \sum_{e=(v,w)\in H^h} (\boldsymbol{f}^p(e_1) - \boldsymbol{f}^p(e_2)) \right| \\
&\leq \epsilon_d^p + \max_{w\in V^h\setminus\{s,t\}} \sum_{e=(v,w)\in H^h} |\boldsymbol{f}^p(e_1) - \boldsymbol{f}^p(e_2)| \\
&\overset{(4)}{\leq} \epsilon_d^p + |H^h|\epsilon_d^p \\
&\overset{(5)}{\leq} |E^h|\epsilon_d^p.
\end{aligned}
\tag{32}
$$

For step (1), we separate homologous edges from other edges that are incident to vertex w. For step (2), we apply the rule of mapping $\boldsymbol{f}^p$ back to $\boldsymbol{f}^h$. For step (3), an example of a vertex $w \in V^h\setminus\{s,t\}$ is illustrated in Figure 3, where w has an incoming homologous edge (v,w) and an outgoing homologous edge (w,u). We apply Eq. (30) in Definition 4.13 with respect to vertex w, so that we can replace the sum of outgoing flows of w by the sum of its incoming flows with an error in demand of G^p being introduced, i.e.,

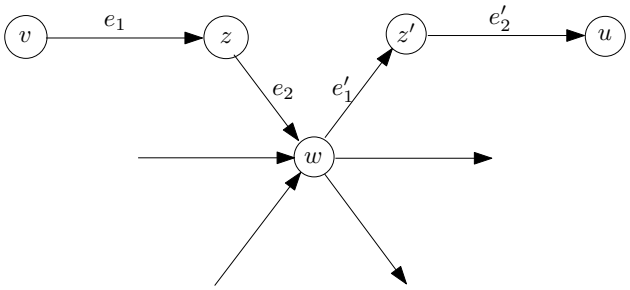

Figure 3: An example of the incoming and outgoing flows of vertex $w \in V^h\setminus\{s,t\}$ when get reduced to G^p. Suppose in G^p, $e = (v,w), e' = (w,u)$ are homologous edges incident to w, and the rest edges incident to w are non-homologous.

$$
\left| \left(\sum_{e=(w,u)\in H^h} \boldsymbol{f}^p(e_1) + \sum_{(w,u)\in E^h\setminus H^h} \boldsymbol{f}^p(w,u) \right) - \left(\sum_{e=(v,w)\in H^h} \boldsymbol{f}^p(e_2) + \sum_{(v,w)\in E^h\setminus H^h} \boldsymbol{f}^p(v,w) \right) \right| \leq \epsilon_d^p.
$$

For step (4), we apply Eq. (30) again with respect to the new inserted vertex. Since the new inserted vertex is only incident to e_1 and e_2, we have

$$|\boldsymbol{f}^p(e_1) - \boldsymbol{f}^p(e_2)| \leq \epsilon_d^p.$$

For step (5), we utilize $|E^h| - |H^h| \geq 1$, i.e., there are more than one non-homologous edges in G^h. As we set in the reduction that $\epsilon_d^p = \frac{\epsilon_d^h}{|E^h|}$, then we have

$$\tau_d^h \leq |E^h| \frac{\epsilon_d^h}{|E^h|} = \epsilon_d^h.$$

3. Error in homology.
   By Eq. (26) in Definition 4.10, the error in homology for a homologous edge set of size k that we can achieve is computed as

$$\begin{aligned}(k-1)\tau_h^h &= \max_{\substack{e,e' \in H_i \in \mathcal{H}^h \\ |H_i|=k}} \left| \boldsymbol{f}^h(e) - \boldsymbol{f}^h(e') \right| \\ &= \max_{\substack{e,e' \in H_i \in \mathcal{H}^h \\ |H_i|=k}} \left| \boldsymbol{f}^p(e_1) - \boldsymbol{f}^p(e_1') \right| \\ &\leq (k-1)\epsilon_h^p.\end{aligned} \tag{33}$$

   where the last inequality comes from a reverse process of telescoping such that the error in pair homology is accumulated for at most $k-1$ times, because H_i is reduced to $k-1$ pair homologous edge sets.

   As we set in the reduction that $\epsilon_h^p = \epsilon_h^h$, then we have

$$\tau_h^h \leq \epsilon_h^h.$$

To summarize, if $\boldsymbol{f}^p$ is a solution to the FPHFA instance with the error $\epsilon_l^p, \epsilon_u^p, \epsilon_d^p, \epsilon_h^p$ as set in the reduction, we can map it back to a solution $\boldsymbol{f}^h$ to the FHFA instance with its error target achieved. □

4.6 FPHF(A) to SFF(A)

4.6.1 FPHF to SFF

We show the reduction from an FPHF instance $(G^p, F^p, \mathcal{H}^p, \boldsymbol{u}^p, s, t)$ to an SFF instance $(G^s, F^s, S_1, S_2, \boldsymbol{u}^s, s_1, t_1, s_2, t_2)$. Assume that $\{e, \hat{e}\} \in \mathcal{H}^p$ is an arbitrary homologous edge set in G^p. As shown in Figure 4, we map $\{e, \hat{e}\}$ in G^p to a gadget consisting edges $\{e_1, e_2, e_3, e_4, e_5 = \hat{e}_3, \hat{e}_1, \hat{e}_2, \hat{e}_4, \hat{e}_5\}$ in G^s. The key idea to remove the homologous requirement is to introduce a second commodity between a source-sink pair (s_2, t_2). Concretely, we impose the fixed flow constraints on $(e_4, \hat{e}_4)$, the selective constraint of commodity 1 on $(e_1, e_2, \hat{e}_1, \hat{e}_2)$, and the selective constraint of commodity 2 on $(e_3, e_5/\hat{e}_3, \hat{e}_5)$. Then, there is a flow of commodity 2 that routes through the directed path $e_3 \to e_4 \to e_5/\hat{e}_3 \to \hat{e}_4 \to \hat{e}_5$, and a flow of commodity 1 through paths $e_1 \to e_4 \to e_2$ and $\hat{e}_1 \to \hat{e}_4 \to \hat{e}_2$. The fixed flow constraint on $(e_4, \hat{e}_4)$ forces the flow of commodity 1 through the two paths to be equal, since the flows of commodity 2 on $(e_4, \hat{e}_4)$ are equal. Thus, the homologous requirement for edge e and $\hat{e}$ is simulated. In addition, as no reduction is performed on

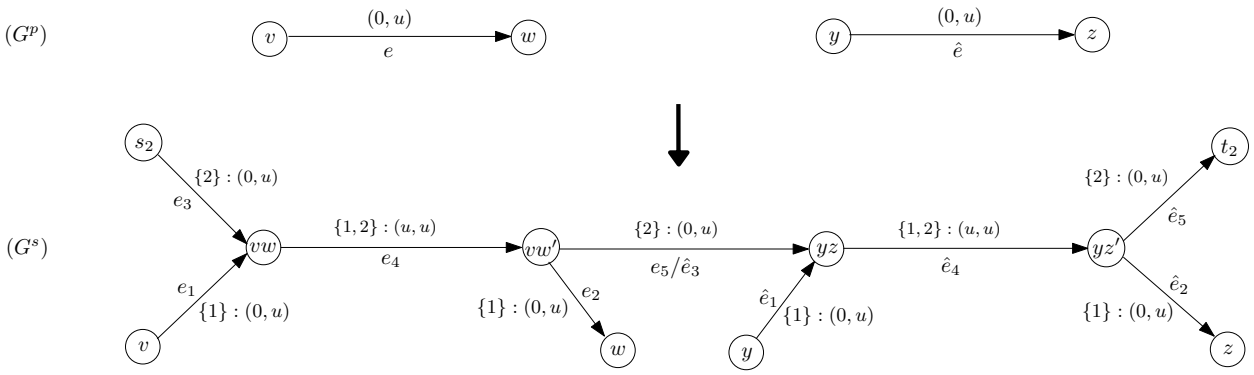

Figure 4: The reduction from FPHF to SFF.

non-homologous edges in G^p, we trivially copy these edges to G^s, and restrict these edges to be selective for commodity 1.

If an SFF solver returns $\boldsymbol{f}^s$ for the SFF instance $(G^s, F^s, S_1, S_2, \boldsymbol{u}^s, s_1, t_1, s_2, t_2)$, then we return $\boldsymbol{f}^p$ for the FPHF instance $(G^p, F^p, \mathcal{H}^p, \boldsymbol{u}^p, s, t)$ by the following method: for each pair of homologous edges e and $\hat{e}$, we set $\boldsymbol{f}^p(e) = \boldsymbol{f}^s_1(e_1), \boldsymbol{f}^p(\hat{e}) = \boldsymbol{f}^s_1(\hat{e}_1)$; for each non-homologous edge e', we set $\boldsymbol{f}^p(e') = \boldsymbol{f}^s_1(e')$. Note that $\boldsymbol{f}^s$ is a two-commodity flow[8] while $\boldsymbol{f}^p$ is a single-commodity flow. If the SFF solver returns "infeasible" for the SFF instance, then we return "infeasible" for the FPHF instance.

Lemma 4.15 (FPHF to SFF)**.** *Given an* FPHF *instance* $(G^p, F^p, \mathcal{H}^p = (H_1, \cdots, H_p), \boldsymbol{u}^p, s, t)$*, we can construct, in time* $O(|E^p|)$*, an* SFF *instance* $(G^s, F^s, S_1, S_2, \boldsymbol{u}^s, s_1, t_1, s_2, t_2)$ *such that*

$$|V^s| = |V^p| + 4p + 2, \qquad |E^s| = |E^p| + 7p, \qquad |F^s| = |F^p| + 2p,$$

$$|S_1| = |E^p| + 2p, \qquad |S_2| = 3p, \qquad \|\boldsymbol{u}^s\|_{max} = \|\boldsymbol{u}^p\|_{max},$$

and if the FPHF *instance has a solution, then the* SFF *instance has a solution.*

Proof. According to the reduction described above, from any solution $\boldsymbol{f}^p$ to the FPHF instance, it is easy to derive a solution $\boldsymbol{f}^s$ to the SFF instance. Concretely, we define a feasible flow $\boldsymbol{f}^s$ as follows:

- For any pair of homologous edge $\{e, \hat{e}\} \in \mathcal{H}^p$, in its corresponding gadget in G^s, we set

  $$\boldsymbol{f}^s_1(e_1) = \boldsymbol{f}^s_1(e_4) = \boldsymbol{f}^s_1(e_2) = \boldsymbol{f}^p(e) = \boldsymbol{f}^p(\hat{e}) = \boldsymbol{f}^s_1(\hat{e}_1) = \boldsymbol{f}^s_1(\hat{e}_4) = \boldsymbol{f}^s_1(\hat{e}_2) \le u,$$

  where $u = \boldsymbol{u}^p(e) = \boldsymbol{u}^p(\hat{e})$, and set

  $$\boldsymbol{f}^s_2(e_1) = \boldsymbol{f}^s_2(e_2) = \boldsymbol{f}^s_2(\hat{e}_1) = \boldsymbol{f}^s_2(\hat{e}_2) = 0.$$

  It is obvious that $\boldsymbol{f}^s$ satisfies the selective constraint and the capacity constraint on edges $e_1, e_2, \hat{e}_1, \hat{e}_2$, and satisfies conservation of flows for commodity 1 on vertices vw, vw', yz, yz'.

  Moreover, we set

  $$\boldsymbol{f}^s_2(e_3) = \boldsymbol{f}^s_2(e_4) = \boldsymbol{f}^s_2(e_5) = \boldsymbol{f}^s_2(\hat{e}_4) = \boldsymbol{f}^s_2(\hat{e}_5) = u - \boldsymbol{f}^p(e) \le u,$$

  and

  $$\boldsymbol{f}^s_1(e_3) = \boldsymbol{f}^s_1(e_4) = \boldsymbol{f}^s_1(e_5) = \boldsymbol{f}^s_1(\hat{e}_4) = \boldsymbol{f}^s_1(\hat{e}_5) = 0.$$

[8] We denote $\boldsymbol{f} = \boldsymbol{f}_1 + \boldsymbol{f}_2$ for two-commodity flows.

Then it is obvious that $\boldsymbol{f}^s$ satisfies the selective constraint and the capacity constraint on edges $e_3, e_5, \hat{e}_5$, and satisfies the flow conservation constraint of commodity 2 on vertices vw, vw', yz, yz'.

It remains to verify if $\boldsymbol{f}^s$ satisfies the fixed flow constraint on edges $e_4, \hat{e}_4$. According to the above constructions, we have (we abuse the notation to also let $\boldsymbol{f}^s = \boldsymbol{f}^s_1 + \boldsymbol{f}^s_1$)

$$\boldsymbol{f}^s(e_4) = \boldsymbol{f}^s_1(e_4) + \boldsymbol{f}^s_2(e_4) = \boldsymbol{f}^p(e) + (u - \boldsymbol{f}^p(e)) = u,$$

$$\boldsymbol{f}^s(\hat{e}_4) = \boldsymbol{f}^s_1(\hat{e}_4) + \boldsymbol{f}^s_2(\hat{e}_4) = \boldsymbol{f}^p(\hat{e}) + (u - \boldsymbol{f}^p(\hat{e})) = u.$$

- For any non-homologous edge $e' \in E^p$, we also have $e' \in E^s$ since no reduction is made on this edge, and we copy it trivially to G^s. We set

$$\boldsymbol{f}^s_1(e') = \boldsymbol{f}^p(e'), \qquad \boldsymbol{f}^s_2(e') = 0.$$

  Since $\boldsymbol{f}^p$ is a feasible flow in G^p, it is easy to check that $\boldsymbol{f}^s$ also satisfies the selective constraint for commodity 1 and the capacity constraint on non-homologous edges, as well as conservation of flows on vertices incident to non-homologous edges.

To conclude, $\boldsymbol{f}^s$ is a feasible flow to the SFF instance.

Now, we track the change of problem size after reduction. Based on the reduction method, given an FPHF instance with $|V^p|$ vertices, $|E^p|$ edges (including $|F^p|$ fixed flow edges, and p pairwise homologous edge sets $\mathcal{H}^p = \{H_1, \cdots, H_p\}$), we can compute the size of the reduced SFF instance as follows.

1. $|V^s|$ vertices. To bound it, first, all vertices in V^p are maintained. Then, for each pair of homologous edges (v, w) and (y, z), 4 auxiliary vertices vw, vw', yz, yz' are added. And finally, a source-sink pair of commodity 2 (s_2, t_2) is added. Therefore, we have

$$|V^s| = |V^p| + 4p + 2.$$

2. $|E^s|$ edges. Since non-homologous edges are maintained, and each pair of homologous edges is replaced by a gadget with 9 edges, thus we have

$$|E^s| = (|E^p| - 2p) + 9p = |E^p| + 7p.$$

3. $|F^s|$ fixed flow edges. First, all fixed flow edges in F^p are maintained since by Def. 2.13, fixed flow edges and homologous edges are disjoint, thus no reduction is made. Then, for each pair of homologous edges as shown in Figure 4, the reduction generate 2 fixed flow edges (i.e., $e_4, \hat{e}_4$). Hence, we have

$$|F^s| = |F^p| + 2p.$$

4. $|S_i|$ edges that select the i-th commodity. First, all non-homologous edges in E^p are selective for the commodity 1. Then, for each pair of homologous edges as shown in Figure 4, the reduction generates 4 edges selecting commodity 1 (i.e., $e_1, e_2, \hat{e}_1, \hat{e}_2$) and 3 edges selecting commodity 2 (i.e., $e_3, e_5/\hat{e}_3, \hat{e}_5$). Thus, we have

$$|S_1| = (|E^p| - 2p) + 4p = |E^p| + 2p, \qquad |S_2| = 3p.$$

5. The maximum edge capacity is bounded by

$$\|\boldsymbol{u}^s\|_{max} = \|\boldsymbol{u}^p\|_{max}\,.$$

   First, capacity of non-homologous edges is unchanged in G^s since no reduction is made. Then, for each pair of homologous edges $\{e, \hat{e}\}$ with capacity $\boldsymbol{u}^p(e)(= \boldsymbol{u}^p(\hat{e}))$, the capacity of the corresponding 9 edges are either selective edges with capacity $\boldsymbol{u}^p(e)$ or fixed flow edges with fixed flow $\boldsymbol{u}^p(e)$.

To estimate the reduction time, first, it takes constant time to reduce each pair of homologous edges since only a constant number of vertices and edges are added. Also, it takes constant time to copy each of the rest non-homologous edges, then the reduction of this step takes $O(|E^p|)$ time in total. □

4.6.2 FPHFA to SFFA

The above lemma shows the reduction between exactly solving an FPHF instance and exactly solving an SFF instance. Next, we generalize the case with exact solutions to the case that allows approximate solutions. First of all, we give a definition of the approximate version of SFF.

Definition 4.16 (SFF Approximate Problem (SFFA))**.** An SFFA instance is given by an SFF instance $(G, F, S_1, S_2, \boldsymbol{u}, s_1, t_1, s_2, t_2)$ as in Definition 2.11, and error parameters $\epsilon_l, \epsilon_u, \epsilon_{d1}, \epsilon_{d2}, \epsilon_{t1}, \epsilon_{t2} \in [0, 1]$, which we collect in a tuple $(G, F, S_1, S_2, \boldsymbol{u}, s_1, t_1, s_2, t_2, \epsilon_l, \epsilon_u, \epsilon_{d1}, \epsilon_{d2}, \epsilon_{t1}, \epsilon_{t2})$. We say an algorithm solves the SFFA problem, if, given any SFFA instance, it returns a pair of flows $\boldsymbol{f}_1, \boldsymbol{f}_2 \geq \boldsymbol{0}$ that satisfies

$$(1-\epsilon_l)\boldsymbol{u}(e) \leq \boldsymbol{f}_1(e) + \boldsymbol{f}_2(e) \leq (1+\epsilon_u)\boldsymbol{u}(e), \quad \forall e \in F \tag{34}$$

$$0 \leq \boldsymbol{f}_1(e) + \boldsymbol{f}_2(e) \leq (1+\epsilon_u)\boldsymbol{u}(e), \quad \forall e \in E\backslash F \tag{35}$$

$$\left| \sum_{u:(u,v)\in E} \boldsymbol{f}_i(u,v) - \sum_{w:(v,w)\in E} \boldsymbol{f}_i(v,w) \right| \leq \epsilon_{di}, \quad \forall v \in V\backslash\{s_i, t_i\},\ i \in \{1,2\} \tag{36}$$

$$\sum_{u:(s_{\bar{i}},w)\in E} \boldsymbol{f}_i(s_{\bar{i}},w) = 0, \sum_{u:(u,t_{\bar{i}})\in E} \boldsymbol{f}_i(u,t_{\bar{i}}) \leq \epsilon_{di}, \quad \bar{i} = \{1,2\}\backslash i \tag{37}$$

$$\boldsymbol{f}_{\bar{i}}(e) \leq \epsilon_{ti}, \quad \forall e \in S_i,\ \bar{i} = \{1,2\}\backslash i \tag{38}$$

or it correctly declares that the associated SFF instance is infeasible. We refer to the error in (34) and (35) as error in congestion, error in (36) and (37) as error in demand, and error in (38) as error in type.

We can use the same reduction and solution mapping method in the exact case to the approximate case.

Lemma 4.17 (FPHFA to SFFA)**.** *Given an* FPHFA *instance* $(G^p, F^p, \mathcal{H}^p = (H_1, \cdots, H_p), \boldsymbol{u}^p, s, t, \epsilon_l^p, \epsilon_u^p, \epsilon_d^p, \epsilon_h^p)$*, we can construct, in* $O(|E^p|)$ *time, an* SFFA *instance* $(G^s, F^s, S_1, S_2, \boldsymbol{u}^s, s_1, t_1, s_2, t_2, \epsilon_l^s, \epsilon_u^s, \epsilon_{d1}^s, \epsilon_{d2}^s, \epsilon_{t1}^s, \epsilon_{t2}^s)$ *such that*

$$|V^s| = |V^p| + 4p + 2, \qquad |E^s| = |E^p| + 7p, \qquad |F^s| = |F^p| + 2p,$$

$$|S_1| = |E^p| + 2p, \qquad |S_2| = 3p, \qquad \|\boldsymbol{u}^s\|_{max} = \|\boldsymbol{u}^p\|_{max}\,,$$

and

$$\epsilon_l^s = \min\left\{\frac{\epsilon_l^p}{2},\quad \frac{\epsilon_h^p}{20\,\|\boldsymbol{u}^p\|_{max}}\right\},\qquad \epsilon_u^s = \min\left\{\epsilon_u^p,\quad \frac{\epsilon_h^p}{20\,\|\boldsymbol{u}^p\|_{max}}\right\},$$

$$\epsilon_{d1}^s = \min\left\{\frac{\epsilon_d^p}{6|E^p|},\quad \frac{\epsilon_h^p}{10}\right\},\qquad \epsilon_{d2}^s = \frac{\epsilon_h^p}{10},$$

$$\epsilon_{t1}^s = \min\left\{\frac{\epsilon_l^p}{2}, \frac{\epsilon_h^p}{10}\right\},\qquad \epsilon_{t2}^s = \min\left\{\frac{\epsilon_d^p}{6|E^p|},\quad \frac{\epsilon_h^p}{10}\right\}.$$

If the FPHF *instance* $(G^p, F^p, \mathcal{H}^p, \boldsymbol{u}^p, s, t)$ *has a solution, then the* SFF *instance* $(G^s, F^s, S_1, S_2, \boldsymbol{u}^s, s_1, t_1, s_2, t_2)$ *has a solution. Furthermore, if* $\boldsymbol{f}^s$ *is a solution to the* SFFA *instance, then in time* $O(|E^p|)$*, we can compute a solution* $\boldsymbol{f}^p$ *to the* FPHFA *instance.*

Proof. Since we use the same reduction method in the exact case to the approximate case, the conclusions in Lemma 4.15 also apply here, including the reduction time, problem size, and that SFF has a feasible solution when FPHF has one. It remains to show the solution mapping time, as well as how the problem error changes by mapping an approximate solution to SFFA back to an approximate solution to FPHFA.

Based on the solution mapping method described above, it takes constant time to set the value of each homologous or non-homologous entry of $\boldsymbol{f}^p$, and $\boldsymbol{f}^p$ has $|E^p|$ entries, such a solution mapping takes $O(|E^p|)$ time.

Now, we conduct an error analysis. We claim that the mapping will generate three types of error: (1) error in congestion; (2) error in demand; (3) error in pair homology.

1. Error in congestion.
   We first track the error of the upper bound of capacity. For any pair of homologous edges e and $\hat{e}$ with capacity $\boldsymbol{u}^p(e) = \boldsymbol{u}^p(\hat{e})$, the maximum flow to be routed in e and $\hat{e}$ is

   $$(1+\tau_u^p)\boldsymbol{u}^p(e) = \boldsymbol{f}_1^s(e_1) \le \boldsymbol{f}^s(e_1) \le (1+\epsilon_u^s)\boldsymbol{u}^s(e_1) = (1+\epsilon_u^s)\boldsymbol{u}^p(e),$$

   and

   $$(1+\tau_u^p)\boldsymbol{u}^p(\hat{e}) = \boldsymbol{f}_1^s(\hat{e}_1) \le \boldsymbol{f}^s(\hat{e}_1) \le (1+\epsilon_u^s)\boldsymbol{u}^s(\hat{e}_1) = (1+\epsilon_u^s)\boldsymbol{u}^p(\hat{e}),$$

   thus,

   $$\tau_u^p \le \epsilon_u^s. \tag{39}$$

   For any non-homologous edge e', then e' is a fixed flow edge with fixed flow $\boldsymbol{u}^p(e')$, the maximum flow to be routed in e' is

   $$(1+\tau_u^p)\boldsymbol{u}^p(e') = \boldsymbol{f}_1^s(e') \le \boldsymbol{f}^s(e') \le (1+\epsilon_u^s)\boldsymbol{u}^s(e') = (1+\epsilon_u^s)\boldsymbol{u}^p(e'),$$

   thus again,

   $$\tau_u^p \le \epsilon_u^s.$$

   Next, we track the error of the lower bound of capacity. And we only need to take fixed flow edges (thus non-homologous edges) into consideration. For any fixed flow edge $e' \in F^p$, as it is copied to an edge selecting commodity 1 in G^s, then the minimum flow to be routed in e'

can be lower bounded by

$$\begin{aligned}
(1-\tau_l^p)\boldsymbol{u}^p(e') &= \boldsymbol{f}_1^s(e') \\
&\geq \boldsymbol{f}^s(e') - \epsilon_{t1}^s && \text{Because of error in type in } G^s \\
&\geq (1-\epsilon_l^s)\boldsymbol{u}^s(e') - \epsilon_{t1}^s && \text{Because of error in congestion in } G^s \\
&= (1-\epsilon_l^s)\boldsymbol{u}^p(e') - \epsilon_{t1}^s && \text{Because } \boldsymbol{u}^p(e') = \boldsymbol{u}^s(e') \\
&= \left(1 - \epsilon_l^s - \frac{\epsilon_{t1}^s}{\boldsymbol{u}^p(e')}\right)\boldsymbol{u}^p(e') \\
&\geq (1 - \epsilon_l^s - \epsilon_{t1}^s)\,\boldsymbol{u}^p(e') && \boldsymbol{u}^p(e') \geq 1 \text{ because of integer capacities}
\end{aligned}$$

Thus, we have

$$\tau_l^p \leq \epsilon_l^s + \epsilon_{t1}^s. \tag{40}$$

2. Error in demand.
For simplicity, we denote $H^p = \bigcup_{i\in[p]} H_i$ as the set of all homologous edges. By Eq. (30) in Definition 4.13, the error in demand that we can achieve for vertices other than s, t in G^p is computed as

$$\begin{aligned}
\tau_d^p &= \max_{w\in V^p\setminus\{s,t\}} \left| \sum_{(v,w)\in E^p} \boldsymbol{f}^p(v,w) - \sum_{(w,u)\in E^p} \boldsymbol{f}^p(w,u) \right| \\
&\overset{(1)}{=} \max_{w\in V^p\setminus\{s,t\}} \left| \left(\sum_{(v,w)\in H^p} \boldsymbol{f}^p(v,w) + \sum_{(v,w)\in E^p\setminus H^p} \boldsymbol{f}^p(v,w) \right) - \left(\sum_{(w,u)\in H^p} \boldsymbol{f}^p(w,u) + \sum_{(w,u)\in E^p\setminus H^p} \boldsymbol{f}^p(w,u) \right) \right| \\
&\overset{(2)}{=} \max_{w\in V^p\setminus\{s,t\}} \left| \left(\sum_{e=(v,w)\in H^p} \boldsymbol{f}_1^s(e_1) + \sum_{(v,w)\in E^p\setminus H^p} \boldsymbol{f}_1^s(v,w) \right) - \left(\sum_{e=(w,u)\in H^p} \boldsymbol{f}_1^s(e_1) + \sum_{(w,u)\in E^p\setminus H^p} \boldsymbol{f}_1^s(w,u) \right) \right| \\
&\overset{(3)}{\leq} \epsilon_{d1}^s + \max_{w\in V^p\setminus\{s,t\}} \left| \left(\sum_{e=(v,w)\in H^p} \boldsymbol{f}_1^s(e_1) + \sum_{(v,w)\in E^p\setminus H^p} \boldsymbol{f}_1^s(v,w) \right) - \left(\sum_{e=(v,w)\in H^p} \boldsymbol{f}_1^s(e_2) + \sum_{(v,w)\in E^p\setminus H^p} \boldsymbol{f}_1^s(v,w) \right) \right| \\
&= \epsilon_{d1}^s + \max_{w\in V^p\setminus\{s,t\}} \left| \sum_{e=(v,w)\in H^p} \boldsymbol{f}_1^s(e_1) - \sum_{e=(v,w)\in H^p} \boldsymbol{f}_1^s(e_2) \right| \\
&= \epsilon_{d1}^s + \max_{w\in V^p\setminus\{s,t\}} \left| \sum_{e=(v,w)\in H^p} (\boldsymbol{f}_1^s(e_1) - \boldsymbol{f}_1^s(e_2)) \right| \\
&\leq \epsilon_{d1}^s + \max_{w\in V^p\setminus\{s,t\}} \sum_{e=(v,w)\in H^p} |\boldsymbol{f}_1^s(e_1) - \boldsymbol{f}_1^s(e_2)|\,.
\end{aligned} \tag{41}$$

For step (1), we separate homologous edges from non-homologous edges that are incident to vertex w. For step (2), we apply the rule of mapping $\boldsymbol{f}^s$ back to $\boldsymbol{f}^p$. For step (3), we apply Eq. (36) in Definition 4.16 with respect to vertex w, so that we replace the sum of outgoing flows of w by the sum of its incoming flows with an error in demand for commodity 1 in G^s being introduced, i.e.,

$$\left| \left(\sum_{e=(w,u)\in H^p} \boldsymbol{f}_1^s(e_1) + \sum_{(w,u)\in E^p\setminus H^p} \boldsymbol{f}_1^s(w,u) \right) - \left(\sum_{e=(v,w)\in H^p} \boldsymbol{f}_1^s(e_2) + \sum_{(v,w)\in E^p\setminus H^p} \boldsymbol{f}_1^s(v,w) \right) \right| \leq \epsilon_{d1}^s.$$

Now, we try to bound $|\boldsymbol{f}_1^s(e_1) - \boldsymbol{f}_1^s(e_2)|$. We apply Eq. (36) again with respect to vertex vw. Since vw is only incident to e_1, e_3, e_4, we have

$$|\boldsymbol{f}_1^s(e_1) + \boldsymbol{f}_1^s(e_3) - \boldsymbol{f}_1^s(e_4)| \leq \epsilon_{d1}^s. \tag{42}$$

Similarly, by error in demand of vertex vw', we have

$$|\boldsymbol{f}_1^s(e_2) + \boldsymbol{f}_1^s(e_5) - \boldsymbol{f}_1^s(e_4)| \leq \epsilon_{d1}^s. \tag{43}$$

Combining Eq. (42) and Eq. (43), we have

$$|(\boldsymbol{f}_1^s(e_1) + \boldsymbol{f}_1^s(e_3)) - (\boldsymbol{f}_1^s(e_2) + \boldsymbol{f}_1^s(e_5))| \leq 2\epsilon_{d1}^s.$$

Moreover, we can also lower bound its left hand side by

$$\begin{aligned}|(\boldsymbol{f}_1^s(e_1) + \boldsymbol{f}_1^s(e_3)) - (\boldsymbol{f}_1^s(e_2) + \boldsymbol{f}_1^s(e_5))| &= |(\boldsymbol{f}_1^s(e_1) - \boldsymbol{f}_1^s(e_2)) + (\boldsymbol{f}_1^s(e_3) - \boldsymbol{f}_1^s(e_5))| \\ &\geq |\boldsymbol{f}_1^s(e_1) - \boldsymbol{f}_1^s(e_2)| - |\boldsymbol{f}_1^s(e_3) - \boldsymbol{f}_1^s(e_5)|\,.\end{aligned}$$

By rearranging, $|\boldsymbol{f}_1^s(e_1) - \boldsymbol{f}_1^s(e_2)|$ can be upper bounded by

$$\begin{aligned}&|\boldsymbol{f}_1^s(e_1) - \boldsymbol{f}_1^s(e_2)| \\ \leq\, &|\boldsymbol{f}_1^s(e_3) - \boldsymbol{f}_1^s(e_5)| + |(\boldsymbol{f}_1^s(e_1) + \boldsymbol{f}_1^s(e_3)) - (\boldsymbol{f}_1^s(e_2) + \boldsymbol{f}_1^s(e_5))| \\ \leq\, &|\boldsymbol{f}_1^s(e_3) - \boldsymbol{f}_1^s(e_5)| + 2\epsilon_{d1}^s \\ \leq\, &\max\{\boldsymbol{f}_1^s(e_3), \boldsymbol{f}_1^s(e_5)\} + 2\epsilon_{d1}^s && \text{Because } \boldsymbol{f}_1^s(e_3), \boldsymbol{f}_1^s(e_5) \geq 0 \\ \leq\, &\epsilon_{t2}^s + 2\epsilon_{d1}^s,\end{aligned} \tag{44}$$

where the last inequality comes from error in type for e_3 and e_5, since they are both selective edges for commodity 2.

Finally, applying the upper bound of $|\boldsymbol{f}_1^s(e_1) - \boldsymbol{f}_1^s(e_2)|$ Eq. (44) back to Eq. (41), we obtain

$$\begin{aligned}\tau_d^p &\leq \epsilon_{d1}^s + |H^p|\,|\boldsymbol{f}_1^s(e_1) - \boldsymbol{f}_1^s(e_2)| \\ &\leq \epsilon_{d1}^s + |H^p|(\epsilon_{t2}^s + 2\epsilon_{d1}^s) \\ &= (2|H^p| + 1)\epsilon_{d1}^s + |H^p|\epsilon_{t2}^s \\ &\leq 3|H^p|(\epsilon_{d1}^s + \epsilon_{t2}^s) && \text{Because } |H^p| \geq 1 \\ &\leq 3|E^p|(\epsilon_{d1}^s + \epsilon_{t2}^s).\end{aligned} \tag{45}$$

3. Error in pair homology.
   By Eq. (31) in Definition 4.13, the error in pair homology that we can achieve is computed as

$$\begin{aligned}\tau_h^p &= \max_{\mathcal{H}^p \ni H_i = \{(v,w),(y,z)\}} |\boldsymbol{f}^p(v,w) - \boldsymbol{f}^p(y,z)| \\ &= \max_{\mathcal{H}^p \ni H_i = \{e=(v,w),\hat{e}=(y,z)\}} |\boldsymbol{f}_1^s(e_1) - \boldsymbol{f}_1^s(\hat{e}_1)|\,,\end{aligned} \tag{46}$$

   where the last equality comes from the rule of mapping $\boldsymbol{f}^s$ back to $\boldsymbol{f}^p$ for homologous edges.

   Now, we try to bound $|\boldsymbol{f}_1^s(e_1) - \boldsymbol{f}_1^s(\hat{e}_1)|$ for an arbitrary pair of homologous edges $e, \hat{e}$ with capacity $\boldsymbol{u}^p(e)(= \boldsymbol{u}^p(\hat{e}))$.

   As e_4 and $\hat{e}_4$ are fixed flow edges, we apply error in congestion as defined in Eq. (34) in Definition 4.16, so that their flow difference can be bounded by

$$|\boldsymbol{f}^s(e_4) - \boldsymbol{f}^s(\hat{e}_4)| \leq (\epsilon_l^s + \epsilon_u^s)\boldsymbol{u}^p(e).$$

   And by error in demand of vertices vw and yz in G^s, we have

$$|\boldsymbol{f}^s(e_1) + \boldsymbol{f}^s(e_3) - \boldsymbol{f}^s(e_4)| \leq \epsilon_{d1}^s + \epsilon_{d2}^s,$$

$$|\boldsymbol{f}^s(\hat{e}_1) + \boldsymbol{f}^s(\hat{e}_3) - \boldsymbol{f}^s(\hat{e}_4)| \le \epsilon^s_{d1} + \epsilon^s_{d2}.$$

Combining the above three inequalities, we have

$$|(\boldsymbol{f}^s(e_1) + \boldsymbol{f}^s(e_3)) - (\boldsymbol{f}^s(\hat{e}_1) + \boldsymbol{f}^s(\hat{e}_3))| \le (\epsilon^s_l + \epsilon^s_u)\boldsymbol{u}^p(e) + 2\epsilon^s_{d1} + 2\epsilon^s_{d2}.$$

Again, we can also lower bound its left hand side by

$$\begin{aligned}
&|(\boldsymbol{f}^s(e_1) + \boldsymbol{f}^s(e_3)) - (\boldsymbol{f}^s(\hat{e}_1) + \boldsymbol{f}^s(\hat{e}_3))| \\
=&|(\boldsymbol{f}^s_1(e_1) + \boldsymbol{f}^s_2(e_1) + \boldsymbol{f}^s_1(e_3) + \boldsymbol{f}^s_2(e_3)) - (\boldsymbol{f}^s_1(\hat{e}_1) + \boldsymbol{f}^s_2(\hat{e}_1) + \boldsymbol{f}^s_1(\hat{e}_3) + \boldsymbol{f}^s_2(\hat{e}_3))| \\
=&|(\boldsymbol{f}^s_1(e_1) - \boldsymbol{f}^s_1(\hat{e}_1)) + (\boldsymbol{f}^s_2(e_1) - \boldsymbol{f}^s_2(\hat{e}_1)) + (\boldsymbol{f}^s_1(e_3) - \boldsymbol{f}^s_1(\hat{e}_3)) + (\boldsymbol{f}^s_2(e_3) - \boldsymbol{f}^s_2(\hat{e}_3))| \\
\ge&|\boldsymbol{f}^s_1(e_1) - \boldsymbol{f}^s_1(\hat{e}_1)| - |(\boldsymbol{f}^s_2(e_1) - \boldsymbol{f}^s_2(\hat{e}_1)) + (\boldsymbol{f}^s_1(e_3) - \boldsymbol{f}^s_1(\hat{e}_3)) + (\boldsymbol{f}^s_2(e_3) - \boldsymbol{f}^s_2(\hat{e}_3))|\,.
\end{aligned}$$

By rearranging, we can upper bound $|\boldsymbol{f}^s_1(e_1) - \boldsymbol{f}^s_1(\hat{e}_1)|$ by

$$\begin{aligned}
&|\boldsymbol{f}^s_1(e_1) - \boldsymbol{f}^s_1(\hat{e}_1)| \\
\le&|(\boldsymbol{f}^s(e_1) + \boldsymbol{f}^s(e_3)) - (\boldsymbol{f}^s(\hat{e}_1) + \boldsymbol{f}^s(\hat{e}_3))| + |(\boldsymbol{f}^s_2(e_1) - \boldsymbol{f}^s_2(\hat{e}_1)) + (\boldsymbol{f}^s_1(e_3) - \boldsymbol{f}^s_1(\hat{e}_3)) + (\boldsymbol{f}^s_2(e_3) - \boldsymbol{f}^s_2(\hat{e}_3))| \\
\le&(\epsilon^s_l + \epsilon^s_u)\boldsymbol{u}^p(e) + 2\epsilon^s_{d1} + 2\epsilon^s_{d2} + |\boldsymbol{f}^s_2(e_1) - \boldsymbol{f}^s_2(\hat{e}_1)| + |\boldsymbol{f}^s_1(e_3) - \boldsymbol{f}^s_1(\hat{e}_3)| + |\boldsymbol{f}^s_2(e_3) - \boldsymbol{f}^s_2(\hat{e}_3)|\,,
\end{aligned}$$

where we have

- $|\boldsymbol{f}^s_2(e_1) - \boldsymbol{f}^s_2(\hat{e}_1)| \le \max\{\boldsymbol{f}^s_2(e_1), \boldsymbol{f}^s_2(\hat{e}_1)\} \le \epsilon^s_{t1}$ because of error in type in G^s and $e_1, \hat{e}_1$ are selective edges for commodity 1;
- $|\boldsymbol{f}^s_1(e_3) - \boldsymbol{f}^s_1(\hat{e}_3)| \le \max\{\boldsymbol{f}^s_1(e_3), \boldsymbol{f}^s_1(\hat{e}_3)\} \le \epsilon^s_{t2}$ because of error in type in G^s and $e_3, \hat{e}_3$ are selective edges for commodity 2;
- $|\boldsymbol{f}^s_2(e_3) - \boldsymbol{f}^s_2(\hat{e}_3)| \le \epsilon^s_{t1} + 2\epsilon^s_{d2}$, which can be obtained directly by symmetry from the bound of $|\boldsymbol{f}^s_1(e_1) - \boldsymbol{f}^s_1(e_2)|$ in Eq. (44).

Putting all together, we have

$$\begin{aligned}
|\boldsymbol{f}^s_1(e_1) - \boldsymbol{f}^s_1(\hat{e}_1)| &\le \boldsymbol{u}^p(e)(\epsilon^s_l + \epsilon^s_u) + 2\epsilon^s_{d1} + 2\epsilon^s_{d2} + \epsilon^{s1}_t + \epsilon^s_{t2} + \epsilon^s_{t1} + 2\epsilon^s_{d2} \\
&\le \|\boldsymbol{u}^p\|_{max}\,(\epsilon^s_l + \epsilon^s_u) + 2\epsilon^s_{d1} + 4\epsilon^s_{d2} + 2\epsilon^s_{t1} + \epsilon^s_{t2}.
\end{aligned}$$

Thus, by Eq. (46), we have

$$\tau^p_h \le \|\boldsymbol{u}^p\|_{max}\,(\epsilon^s_l + \epsilon^s_u) + 2\epsilon^s_{d1} + 4\epsilon^s_{d2} + 2\epsilon^s_{t1} + \epsilon^s_{t2}. \tag{47}$$

Now, we prove that $\boldsymbol{f}^p$ is a solution to the FPHFA instance by showing that the errors achieved by $\boldsymbol{f}^p$ is less than the error target set in the FPHFA instance. Combining Eq. (39), (40), (45), (47), we have

$$\begin{aligned}
&\tau^p_u \le \epsilon^s_u, \\
&\tau^p_l \le \epsilon^s_l + \epsilon^s_{t1}, \\
&\tau^p_d \le 3|E^p|(\epsilon^s_{d1} + \epsilon^s_{t2}), \\
&\tau^p_h \le \|\boldsymbol{u}^p\|_{max}\,(\epsilon^s_l + \epsilon^s_u) + 2\epsilon^s_{d1} + 4\epsilon^s_{d2} + 2\epsilon^s_{t1} + \epsilon^s_{t2}.
\end{aligned}$$

As we set in the reduction that $\epsilon^s_l = \min\left\{\frac{\epsilon^p_l}{2}, \frac{\epsilon^p_h}{20\|\boldsymbol{u}^p\|_{max}}\right\}$, $\epsilon^s_u = \min\left\{\epsilon^p_u, \frac{\epsilon^p_h}{20\|\boldsymbol{u}^p\|_{max}}\right\}$, $\epsilon^s_{d1} = \min\left\{\frac{\epsilon^p_d}{6|E^p|}, \frac{\epsilon^p_h}{10}\right\}$, $\epsilon^s_{d2} = \frac{\epsilon^p_h}{10}$, $\epsilon^s_{t1} = \min\left\{\frac{\epsilon^p_l}{2}, \frac{\epsilon^p_h}{10}\right\}$, $\epsilon^s_{t2} = \min\left\{\frac{\epsilon^p_d}{6|E^p|}, \frac{\epsilon^p_h}{10}\right\}$, thus we have

$$\tau^p_u \le \epsilon^p_u, \qquad \tau^p_l \le \frac{\epsilon^p_l}{2} + \frac{\epsilon^p_l}{2} = \epsilon^p_l,$$

$$\tau_d^p \le 3|E^p| \left(\frac{\epsilon_d^p}{6|E^p|} + \frac{\epsilon_d^p}{6|E^p|} \right) = \epsilon_d^p,$$

$$\tau_h^p \le \|\boldsymbol{u}^p\|_{max} \left(\frac{\epsilon_h^p}{20\,\|\boldsymbol{u}^p\|_{max}} + \frac{\epsilon_h^p}{20\,\|\boldsymbol{u}^p\|_{max}} \right) + \frac{2\epsilon_h^p}{10} + \frac{4\epsilon_h^p}{10} + \frac{2\epsilon_h^p}{10} + \frac{\epsilon_h^p}{10} = \epsilon_h^p.$$

Thus, we can conclude that if $\boldsymbol{f}^s$ is a solution to the SFFA instance with the error $\epsilon_l^s, \epsilon_u^s, \epsilon_{d1}^s, \epsilon_{d2}^s, \epsilon_{t1}^s, \epsilon_{t2}^s$ as set in the reduction, we can map it back to a solution $\boldsymbol{f}^p$ to the FPHFA instance with its error target achieved. □

4.7 SFF(A) to 2CFF(A)

4.7.1 SFF to 2CFF

We show the reduction from an SFF instance $(G^s, F^s, S_1, S_2, \boldsymbol{u}^s, s_1, t_1, s_2, t_2)$ to a 2CFF instance $(G^f, F^f, \boldsymbol{u}^f, s_1, t_1, s_2, t_2)$. Assume that $e \in S_i$ is an arbitrary selective edge for commodity i in G^p. As shown in Figure 5, we map e in G^s to a gadget consisting of edges $\{e_1, e_2, e_3, e_4, e_5\}$ in G^f. Note that a selective edge e can be either a fixed flow edge or a non-fixed flow edge. In Figure 5, $l = u$ if e is a fixed flow edge and $l = 0$ if e is a non-fixed flow edge. Moreover, no reduction is performed on non-selective edges in G^s, and we trivially copy these edges to G^f.

The key idea to remove the selective requirement is utilizing edge directions and the source-sink pair (s_i, t_i) to simulate a selective edge e for commodity i. More specifically, in the gadget, the flow of commodity i routes through three directed paths: (1) $e_1 \to e_4$, (2) $e_5 \to e_2 \to e_4$, (3) $e_5 \to e_3$. The selective requirement is realized because e_4 is the only outgoing edge of xy and only flow of commodity i is allowed in e_4 (since its tail is t_i), thus in e_1. Similarly, e_5 is the only incoming edge of xy' and only flow of commodity i is allowed in e_5, thus in e_3. In addition, to ensure that e_1 and e_3 route the same amount of flow, flows in e_4 and e_5 must be equal by the conservation of flows. Therefore, we impose the fixed flow constraint on e_4 and e_5 by setting the fixed flow to be u. We remove e_2 if e is a fixed flow edge (in which case e_2 has capacity 0), and set capacity of e_2 to be u otherwise (in which case $u - l = u - 0 = u$).

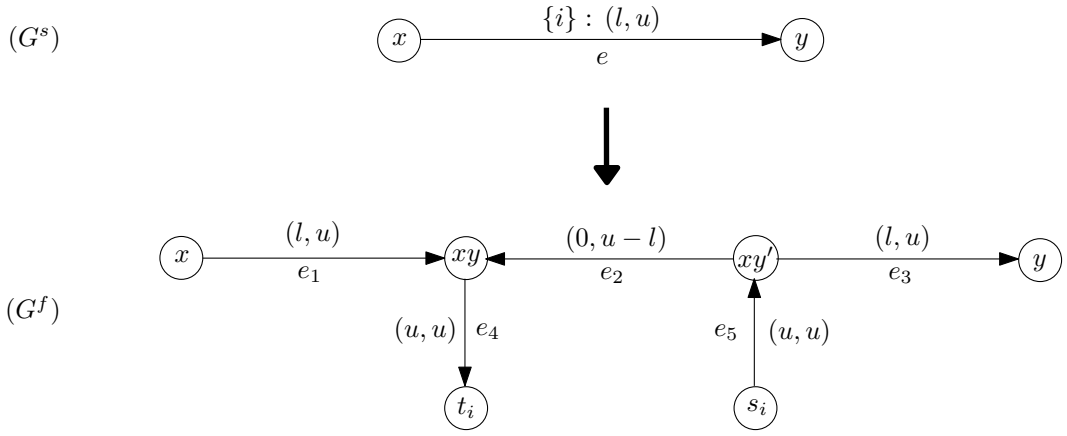

Figure 5: The reduction from SFF to 2CFF. $l = u$ if e is a fixed flow edge, and $l = 0$ if e is a non-fixed flow edge.

If a 2CFF solver returns $\boldsymbol{f}^f$ for the 2CFF instance $(G^f, F^f, \boldsymbol{u}^f, s_1, t_1, s_2, t_2)$, then we return $\boldsymbol{f}^s$ for the SFF instance $(G^s, F^s, S_1, S_2, \boldsymbol{u}^s, s_1, t_1, s_2, t_2)$ by the following method: for any selective edge $e \in S_i$, we set $\boldsymbol{f}_i^s(e) = \boldsymbol{f}_i^f(e_1), \boldsymbol{f}_{\bar{i}}^s(e) = \boldsymbol{f}_{\bar{i}}^f(e_1) = 0, \bar{i} \in \{1, 2\} \backslash i$; and for any non-selective edge

$e \in E^s \backslash (S_1 \bigcup S_2)$, we map back trivially by setting $\boldsymbol{f}_i^s(e) = \boldsymbol{f}_i^f(e), i \in \{1,2\}$. If the 2CFF solver returns "infeasible" for the 2CFF instance, then we return "infeasible" for the SFF instance.

Lemma 4.18 (SFF to 2CFF)**.** *Given an* SFF *instance* $(G^s, F^s, S_1, S_2, \boldsymbol{u}^s, s_1, t_1, s_2, t_2)$*, we can construct, in time* $O(|E^s|)$*, a* 2CFF *instance* $(G^f, F^f, \boldsymbol{u}^f, s_1, t_1, s_2, t_2)$ *such that*

$$|V^f| = |V^s| + 2(|S_1| + |S_2|), \qquad |E^f| = |E^s| + 4(|S_1| + |S_2|), \qquad |F^f| \leq 4(|F^s| + |S_1| + |S_2|),$$

$$\left\| \boldsymbol{u}^f \right\|_{max} = \left\| \boldsymbol{u}^s \right\|_{max},$$

and if the SFF *instance has a solution, then the* 2CFF *instance has a solution.*

Proof. According to the reduction described above, from any solution $\boldsymbol{f}^s$ to the SFF instance, it is easy to derive a solution $\boldsymbol{f}^f$ to the 2CFF instance. Concretely, we define a feasible flow $\boldsymbol{f}^f$ as follows:

- For any selective edge $e \in S_i$, in its corresponding gadget in G^f, we set

$$l \leq \boldsymbol{f}_i^f(e_1) = \boldsymbol{f}_i^f(e_3) = \boldsymbol{f}_i^s(e) \leq u,$$

$$\boldsymbol{f}_i^f(e_4) = \boldsymbol{f}_i^f(e_5) = u,$$

$$0 \leq \boldsymbol{f}_i^f(e_2) = u - \boldsymbol{f}_i^s(e) \leq u - l,$$

  where $l = 0$ if e is an non-fixed flow edge, or $l = u = \boldsymbol{u}^s(e)$ if e is a fixed flow edge. And we set

$$\boldsymbol{f}_{\bar{i}}^f(e_1) = \boldsymbol{f}_{\bar{i}}^f(e_2) = \boldsymbol{f}_{\bar{i}}^f(e_3) = \boldsymbol{f}_{\bar{i}}^f(e_4) = \boldsymbol{f}_{\bar{i}}^f(e_5) = 0, \qquad \bar{i} = \{1,2\} \backslash i.$$

  It is obvious that $\boldsymbol{f}^f$ satisfies the capacity constraint on edges $(e_1, \ldots, e_5)$, and the flow conservation constraint on vertices xy, xy'.

- For any non-selective edge $e' \in E^s \backslash (S_1 \bigcup S_2)$, we also have $e' \in E^f$ since no reduction is made on this edge, and we copy it trivially to G^f. We set

$$\boldsymbol{f}_i^f(e') = \boldsymbol{f}_i^s(e'), \qquad i \in \{1,2\}.$$

  Since $\boldsymbol{f}^s$ is a feasible flow in G^s, it is easy to check that $\boldsymbol{f}^f$ also satisfies the capacity constraint on non-selective edges, as well as the flow conservation constraint on vertices incident to non-selective edges. Moreover, if e' is a fixed flow edge, the fixed flow edge constraint is also satisfied.

To conclude, $\boldsymbol{f}^f$ is a feasible flow to the 2CFF instance.

Now, we track the change of problem size after reduction. Based on the reduction method, given an SFF instance with $|V^s|$ vertices, $|E^s|$ edges (including $|F^s|$ fixed flow edges, and $|S_i|$ edges selecting commodity i), we can compute the size of the reduced 2CFF instance as follows.

1. $|V^f|$ vertices. As all vertices in V^s are maintained, and for each selective edge $(x, y) \in (S_1 \cup S_2)$, 2 auxiliary vertices xy, xy' are added, we have

$$|V^f| = |V^s| + 2(|S_1| + |S_2|).$$

2. $|E^f|$ edges. Since all non-selective edges in $E^s \backslash (S_1 \cup S_2)$ are maintained, and each selective edge $e \in (S_1 \cup S_2)$ is replaced by a gadget with 5 edges $(e_1, \cdots, e_5)$, we have

$$|E^f| = (|E^s| - |S_1| - |S_2|) + 5(|S_1| + |S_2|) = |E^s| + 4(|S_1| + |S_2|).$$

3. $|F^f|$ fixed flow edges. First, all non-selective fixed flow edges $e \in F^s \backslash (S_1 \cup S_2)$ are maintained since no reduction is made. Then, for all selective fixed flow edges $e \in F^s \bigcap (S_1 \cup S_2)$, the reduction generates 4 fixed flow edges (i.e., e_1, e_3, e_4, e_5). Finally, for all non-fixed flow selective edges $e \in (S_1 \cup S_2) \backslash F^s$, the reduction generates 2 fixed flow edges (i.e., e_4, e_5). Hence, we have

$$|F^f| = |F^s \backslash (S_1 \cup S_2)| + 4|F^s \cap (S_1 \cup S_2)| + 2|(S_1 \cup S_2) \backslash F^s| \leq 4|F^s \cup S_1 \cup S_2| \leq 4(|F^s| + |S_1| + |S_2|).$$

4. The maximum edge capacity is bounded by

$$\left\| \boldsymbol{u}^f \right\|_{max} = \left\| \boldsymbol{u}^s \right\|_{max}.$$

   First, capacity of all non-selective edges in $E^s \backslash (S_1 \cup S_2)$ is unchanged in G^f since no reduction is made. Then, for all selective edges $e \in (S_1 \cup S_2)$ with capacity (or fixed flow) $\boldsymbol{u}^s(e)$, capacity of the 5 edges $(e_1, \ldots, e_5)$ in the gadget are with capacity at most $\boldsymbol{u}^s(e)$.

To estimate the reduction time, it is observed that it takes constant time to reduce each selective edge in $S_1 \cup S_2$ since only a constant number of vertices and edges are added. And it takes constant time to copy each of the other non-selective edges in $E^s \backslash (S_1 \cup S_2)$. Thus, the reduction of this step takes $O(|E^s|)$ time. □

4.7.2 SFFA to 2CFFA

The above lemma shows the reduction between exactly solving an SFF instance and exactly solving a 2CFF instance. Next, we generalize the case with exact solutions to the case that allows approximate solutions. First of all, we give a definition of the approximate version of 2CFF.

Definition 4.19 (2CFF Approximate Problem (2CFFA))**.** A 2CFFA instance is given by a 2CFF instance $(G, F, \boldsymbol{u}, s_1, t_1, s_2, t_2)$ as in Definition 2.10, and error parameters $\epsilon_l, \epsilon_u, \epsilon_{d1}, \epsilon_{d2} \in [0, 1]$, which we collect in a tuple $(G, F, \boldsymbol{u}, s_1, t_1, s_2, t_2, \epsilon_l, \epsilon_u, \epsilon_{d1}, \epsilon_{d2})$. We say an algorithm solves the 2CFFA problem, if, given any 2CFFA instance, it returns a pair of flows $\boldsymbol{f}_1, \boldsymbol{f}_2 \geq \boldsymbol{0}$ (denote $\boldsymbol{f} = \boldsymbol{f}_1 + \boldsymbol{f}_2$) that satisfies

$$(1 - \epsilon_l)\boldsymbol{u}(e) \leq \boldsymbol{f}_1(e) + \boldsymbol{f}_2(e) \leq (1 + \epsilon_u)\boldsymbol{u}(e), \quad \forall e \in F \tag{48}$$

$$0 \leq \boldsymbol{f}_1(e) + \boldsymbol{f}_2(e) \leq (1 + \epsilon_u)\boldsymbol{u}(e), \quad \forall e \in E \backslash F \tag{49}$$

$$\left| \sum_{u:(u,v)\in E} \boldsymbol{f}_i(u, v) - \sum_{w:(v,w)\in E} \boldsymbol{f}_i(v, w) \right| \leq \epsilon_{di}, \quad \forall v \in V \backslash \{s_i, t_i\},\ i \in \{1, 2\} \tag{50}$$

$$\sum_{w:(s_{\bar{i}},w)\in E} \boldsymbol{f}_i(s_{\bar{i}}, w) = 0, \quad \sum_{u:(u,t_{\bar{i}})\in E} \boldsymbol{f}_i(u, t_{\bar{i}}) \leq \epsilon_{di}, \quad \bar{i} = \{1, 2\} \backslash i \tag{51}$$

or it correctly declares that the associated 2CFF instance is infeasible. We refer to the error in (48) and (49) as error in congestion, error in (50) and (51) as error in demand.

We can use the same reduction method and solution mapping method in the exact case to the approximate case. Note that, different from the exact case where $\boldsymbol{f}_{\bar{i}}^{s}(e) = \boldsymbol{f}_{\bar{i}}^{f}(e_1) = 0$ if $e \in S_i$, it can be nonzero in the approximate case, which gives rise to error in type in G^s.

Lemma 4.20 (SFFA to 2CFFA)**.** *Given an* SFFA *instance* $(G^s, F^s, S_1, S_2, \boldsymbol{u}^s, s_1, t_1, s_2, t_2, \epsilon_l^s, \epsilon_u^s, \epsilon_{d1}^s, \epsilon_{d2}^s, \epsilon_{t1}^s, \epsilon_{t2}^s)$*, we can construct, in* $O(|E^s|)$ *time, an* 2CFFA *instance* $\left(G^f, F^f, \boldsymbol{u}^f, s_1, t_1, s_2, t_2, \epsilon_l^f, \epsilon_u^f, \epsilon_{d1}^f, \epsilon_{d2}^f\right)$ *such that*

$$|V^f| = |V^s| + 2(|S_1| + |S_2|), \qquad |E^f| = |E^s| + 4(|S_1| + |S_2|), \qquad |F^f| \leq 4(|F^s| + |S_1| + |S_2|),$$

$$\left\|\boldsymbol{u}^f\right\|_{max} = \|\boldsymbol{u}^s\|_{max},$$

and

$$\epsilon_l^f = \min\left\{\epsilon_l^s, \frac{\min\{\epsilon_{d1}^s, \epsilon_{d2}^s\}}{12(|S_1| + |S_2|)\,\|\boldsymbol{u}^s\|_{max}}\right\}, \qquad \epsilon_u^f = \min\left\{\epsilon_u^s, \frac{\min\{\epsilon_{d1}^s, \epsilon_{d2}^s\}}{12(|S_1| + |S_2|)\,\|\boldsymbol{u}^s\|_{max}}\right\},$$

$$\epsilon_{d1}^f = \min\left\{\frac{\min\{\epsilon_{d1}^s, \epsilon_{d2}^s\}}{12(|S_1| + |S_2|)}, \frac{\epsilon_{t2}^s}{2}\right\}, \qquad \epsilon_{d2}^f = \min\left\{\frac{\min\{\epsilon_{d1}^s, \epsilon_{d2}^s\}}{12(|S_1| + |S_2|)}, \frac{\epsilon_{t2}^s}{2}\right\}.$$

If the SFF *instance* $(G^s, F^s, S_1, S_2, \boldsymbol{u}^s, s_1, t_1, s_2, t_2)$ *has a solution, then the* 2CFFA *instance* $\left(G^f, F^f, \boldsymbol{u}^f, s_1, t_1, s_2, t_2\right)$ *has a solution. Furthermore, if* $\boldsymbol{f}^f$ *is a solution to the* 2CFFA *instance, then in time* $O(|E^s|)$*, we can compute a solution* $\boldsymbol{f}^s$ *to the* SFFA *instance.*

Proof. Since we use the same reduction method in the exact case to the approximate case, the conclusions in Lemma 4.18 also apply here, including the reduction time, problem size, and that 2CFF has a feasible solution when SFF has one. It remains to show the solution mapping time, as well as how the problem error changes by mapping an approximate solution to 2CFFA back to an approximate solution to SFFA.

Based on the solution mapping method described above, it takes constant time to set the value of each selective or non-selective entry of $\boldsymbol{f}^s$, and $\boldsymbol{f}^s$ has $|E^s|$ entries, such a solution mapping takes $O(|E^s|)$ time.

Now, we conduct an error analysis. The mapping will generate three types of error: (1) error in congestion; (2) error in demand; (3) error in type.

1. Error in congestion.
   Since we map solution back trivially for non-selective edges in $E^s \backslash (S_1 \cup S_2)$, error in congestion of G^f (i.e., $\epsilon_l^f, \epsilon_u^f$) still holds for G^s. Therefore, we focus on how error in congestion changes when mapping solution back to selective edges in $S_1 \cup S_2$.

   We first track the error of the upper bound of capacity. For any selective edge e,

   $$\sum_{i=\{1,2\}} \boldsymbol{f}_i^s(e) = \sum_{i=\{1,2\}} \boldsymbol{f}_i^f(e_1) \leq (1 + \epsilon_u^f)\boldsymbol{u}^f(e) = (1 + \epsilon_u^f)\boldsymbol{u}^s(e),$$

   thus,

   $$\tau_u^s \leq \epsilon_u^f. \tag{52}$$

   Next, we track the error of the lower bound of capacity. We only need to take selective fixed flow edges into consideration since there is no error of the lower bound of capacity for non-fixed flow edges. If e is a fixed flow edge, then e_1 is also a fixed flow edge with the same

capacity as e. Hence, we can lower bound the flow routed through e by error in congestion of e_1 in G^f.

$$\boldsymbol{f}^s(e) = \boldsymbol{f}^f(e_1) \geq (1 - \epsilon_l^f)\boldsymbol{u}^s(e),$$

and thus

$$\tau_l^s \leq \epsilon_l^f. \tag{53}$$

2. Error in demand.
For simplicity, we denote $S = S_1 \bigcup S_2$. By Eq. (36) in Definition 4.16, error in demand of commodity i that we can achieve for vertices other than s_i, t_i in G^s is computed as

$$\begin{aligned}
\tau_{di}^s &= \max_{y \in V^s \setminus \{s_i, t_i\}} \left| \sum_{(x,y) \in E^s} \boldsymbol{f}_i^s(x,y) - \sum_{(y,z) \in E^s} \boldsymbol{f}_i^s(y,z) \right| \\
&\overset{(1)}{=} \max_{y \in V^s \setminus \{s_i, t_i\}} \left| \left(\sum_{(x,y) \in S} \boldsymbol{f}_i^s(x,y) + \sum_{(x,y) \in E^s \setminus S} \boldsymbol{f}_i^s(x,y) \right) - \left(\sum_{(y,z) \in S} \boldsymbol{f}_i^s(y,z) + \sum_{(y,z) \in E^s \setminus S} \boldsymbol{f}_i^s(y,z) \right) \right| \\
&\overset{(2)}{=} \max_{y \in V^s \setminus \{s_i, t_i\}} \left| \left(\sum_{e=(x,y) \in S} \boldsymbol{f}_i^f(e_1) + \sum_{(x,y) \in E^s \setminus S} \boldsymbol{f}_i^f(x,y) \right) - \left(\sum_{e=(y,z) \in S} \boldsymbol{f}_i^f(e_1) + \sum_{(y,z) \in E^s \setminus S} \boldsymbol{f}_i^f(y,z) \right) \right| \\
&\overset{(3)}{\leq} \epsilon_{di}^f + \max_{y \in V^s \setminus \{s_i, t_i\}} \left| \left(\sum_{e=(x,y) \in S} \boldsymbol{f}_i^f(e_1) + \sum_{(x,y) \in E^s \setminus S} \boldsymbol{f}_i^f(x,y) \right) - \left(\sum_{e=(x,y) \in S} \boldsymbol{f}_i^f(e_3) + \sum_{(x,y) \in E^s \setminus S} \boldsymbol{f}_i^f(x,y) \right) \right| \\
&= \epsilon_{di}^f + \max_{y \in V^s \setminus \{s_i, t_i\}} \left| \sum_{e=(x,y) \in S} \boldsymbol{f}_i^f(e_1) - \sum_{e=(x,y) \in S} \boldsymbol{f}_i^f(e_3) \right| \\
&= \epsilon_{di}^f + \max_{y \in V^s \setminus \{s_i, t_i\}} \left| \sum_{e=(x,y) \in S} (\boldsymbol{f}_i^f(e_1) - \boldsymbol{f}_i^f(e_3)) \right| \\
&\leq \epsilon_{di}^f + \max_{y \in V^s \setminus \{s_i, t_i\}} \sum_{e=(x,y) \in S} \left| \boldsymbol{f}_i^f(e_1) - \boldsymbol{f}_i^f(e_3) \right| \\
&= \epsilon_{di}^f + \max_{y \in V^s \setminus \{s_i, t_i\}} \left\{ \sum_{e=(x,y) \in S_1} \left| \boldsymbol{f}_i^f(e_1) - \boldsymbol{f}_i^f(e_3) \right| + \sum_{\hat{e}=(x,y) \in S_2} \left| \boldsymbol{f}_i^f(\hat{e}_1) - \boldsymbol{f}_i^f(\hat{e}_3) \right| \right\}.
\end{aligned} \tag{54}$$

For step (1), we separate selective edges from non-selective edges that are incident to vertex y. For step (2), we apply the rule of mapping $\boldsymbol{f}^f$ back to $\boldsymbol{f}^s$. For step (3), we apply Eq. (50) in Definition 4.19 with respect to vertex y, so that we can replace the sum of outgoing flows of y by the sum of incoming flows, where an error of ϵ_{di}^f is introduced because of the error in demand of vertex y in G^f, i.e.,

$$\left| \left(\sum_{e=(y,z) \in S} \boldsymbol{f}_i^f(e_1) + \sum_{(y,z) \in E^s \setminus S} \boldsymbol{f}_i^f(y,z) \right) - \left(\sum_{e=(x,y) \in S} \boldsymbol{f}_i^f(e_3) + \sum_{(x,y) \in E^s \setminus S} \boldsymbol{f}_i^f(x,y) \right) \right| \leq \epsilon_{di}^f.$$

Now, we try to bound $\left| \boldsymbol{f}_i^f(e_1) - \boldsymbol{f}_i^f(e_3) \right|$. We apply Eq. (50) again with respect to vertex xy. Since xy is only incident to e_1, e_2, e_4, we have

$$\left| \boldsymbol{f}_i^f(e_1) + \boldsymbol{f}_i^f(e_2) - \boldsymbol{f}_i^f(e_4) \right| \leq \epsilon_{di}^f.$$

Similarly, by the error in demand of vertex xy', we have

$$\left| \boldsymbol{f}_i^f(e_3) + \boldsymbol{f}_i^f(e_2) - \boldsymbol{f}_i^f(e_5) \right| \leq \epsilon_{di}^f.$$

Combining the above two inequalities, we have

$$-2\epsilon_{di}^f + \boldsymbol{f}_i^f(e_4) - \boldsymbol{f}_i^f(e_5) \le \boldsymbol{f}_i^f(e_1) - \boldsymbol{f}_i^f(e_3) \le 2\epsilon_{di}^f + \boldsymbol{f}_i^f(e_4) - \boldsymbol{f}_i^f(e_5),$$

which can be rewritten as

$$\left|\boldsymbol{f}_i^f(e_1) - \boldsymbol{f}_i^f(e_3)\right| \le 2\epsilon_{di}^f + \left|\boldsymbol{f}_i^f(e_4) - \boldsymbol{f}_i^f(e_5)\right|. \tag{55}$$

Wlog, we assume $i = 1$. By symmetry, we can get the same bound for $i = 2$. Eq. (55) can be further bounded by

$$\begin{aligned}
\left|\boldsymbol{f}_1^f(e_1) - \boldsymbol{f}_1^f(e_3)\right| &\le 2\epsilon_{d1}^f + \left|\boldsymbol{f}_1^f(e_4) - \boldsymbol{f}_1^f(e_5)\right| \\
&= 2\epsilon_{d1}^f + \left|(\boldsymbol{f}^f(e_4) - \boldsymbol{f}_2^f(e_4)) - (\boldsymbol{f}^f(e_5) - \boldsymbol{f}_2^f(e_5))\right| \\
&\le 2\epsilon_{d1}^f + \left|\boldsymbol{f}^f(e_4) - \boldsymbol{f}^f(e_5)\right| + \left|\boldsymbol{f}_2^f(e_4) - \boldsymbol{f}_2^f(e_5)\right| \\
&\overset{(1)}{=} 2\epsilon_{d1}^f + \left|\boldsymbol{f}^f(e_4) - \boldsymbol{f}^f(e_5)\right| + \boldsymbol{f}_2^f(e_4) \\
&\overset{(2)}{\le} 2\epsilon_{d1}^f + \left|\boldsymbol{f}^f(e_4) - \boldsymbol{f}^f(e_5)\right| + \epsilon_{d2}^f \\
&\overset{(3)}{\le} 2\epsilon_{d1}^f + (\epsilon_l^f + \epsilon_u^f)\boldsymbol{u}^s(e) + \epsilon_{d2}^f.
\end{aligned} \tag{56}$$

For step (1), we use $\boldsymbol{f}_2^f(e_5) = 0$ because there is no error in demand for s_1 in G^f by Eq. (51) in Definition 4.19. For step (2), we use $\boldsymbol{f}_2^f(e_4) \le \epsilon_{d2}^f$ because of the error in demand for t_1 in G^f again by Eq. (51) in Definition 4.19. For step (3), we use the error in congestion for e_4 and e_5 in G^f.

Next, for flow of commodity 2, i.e., $i = 2$, Eq. (55) can be further bounded as

$$\begin{aligned}
\left|\boldsymbol{f}_2^f(e_1) - \boldsymbol{f}_2^f(e_3)\right| &\le 2\epsilon_{d2}^f + \left|\boldsymbol{f}_2^f(e_4) - \boldsymbol{f}_2^f(e_5)\right| \\
&\overset{(1)}{\le} 2\epsilon_{d2}^f + \epsilon_{d2}^f \\
&= 3\epsilon_{d2}^f.
\end{aligned} \tag{57}$$

For step (1), we use $\boldsymbol{f}_2^f(e_4) \le \epsilon_{d2}^f, \boldsymbol{f}_2^f(e_5) = 0$ with the same reasons as explained before.

By symmetry, we get

$$\left|\boldsymbol{f}_1^f(\hat{e}_1) - \boldsymbol{f}_1^f(\hat{e}_3)\right| \le 3\epsilon_{d1}^f, \tag{58}$$

$$\left|\boldsymbol{f}_2^f(\hat{e}_1) - \boldsymbol{f}_2^f(\hat{e}_3)\right| \le 2\epsilon_{d2}^f + (\epsilon_l^f + \epsilon_u^f)\boldsymbol{u}^s(\hat{e}) + \epsilon_{d1}^f. \tag{59}$$

Finally, applying Eq. (56) and Eq. (58) to Eq. (54), we obtain

$$\begin{aligned}
\tau^s_{d1} &\le \epsilon^f_{d1} + \max_{y\in V^s\setminus\{s_1,t_1\}} \left\{ \sum_{e=(x,y)\in S_1} \left| \boldsymbol{f}^f_1(e_1) - \boldsymbol{f}^f_1(e_3) \right| + \sum_{\hat{e}=(x,y)\in S_2} \left| \boldsymbol{f}^f_1(\hat{e}_1) - \boldsymbol{f}^f_1(\hat{e}_3) \right| \right\} \\
&\le \epsilon^f_{d1} + \max_{y\in V^s\setminus\{s_1,t_1\}} \left\{ \sum_{e=(x,y)\in S_1} \left(2\epsilon^f_{d1} + (\epsilon^f_l + \epsilon^f_u)\boldsymbol{u}^s(e) + \epsilon^f_{d2} \right) + \sum_{\hat{e}=(x,y)\in S_2} 3\epsilon^f_{d1} \right\} \\
&\le \epsilon^f_{d1} + |S_1|(2\epsilon^f_{d1} + \epsilon^f_{d2} + (\epsilon^f_l + \epsilon^f_u)\left\|\boldsymbol{u}^s\right\|_{max}) + |S_2|\cdot 3\epsilon^f_{d1} \\
&= (2|S_1| + 3|S_2| + 1)\epsilon^f_{d1} + |S_1|\epsilon^f_{d2} + |S_1|\left\|\boldsymbol{u}^s\right\|_{max}(\epsilon^f_l + \epsilon^f_u) \\
&\le 3(|S_1| + |S_2|)\left(\epsilon^f_{d1} + \epsilon^f_{d2} + \left\|\boldsymbol{u}^s\right\|_{max}(\epsilon^f_l + \epsilon^f_u) \right),
\end{aligned} \tag{60}$$

where we use $|S_1| \ge 1$ in the last inequality.

Similarly, applying Eq. (57) and Eq. (59) to Eq. (54), we obtain

$$\begin{aligned}
\tau^s_{d2} &\le \epsilon^f_{d2} + \max_{y\in V^s\setminus\{s_2,t_2\}} \left\{ \sum_{e=(x,y)\in S_1} \left| \boldsymbol{f}^f_2(e_1) - \boldsymbol{f}^f_2(e_3) \right| + \sum_{\hat{e}=(x,y)\in S_2} \left| \boldsymbol{f}^f_2(\hat{e}_1) - \boldsymbol{f}^f_2(\hat{e}_3) \right| \right\} \\
&\le \epsilon^f_{d2} + \max_{y\in V^s\setminus\{s_2,t_2\}} \left\{ \sum_{e=(x,y)\in S_1} 3\epsilon^f_{d2} + \sum_{\hat{e}=(x,y)\in S_2} \left(2\epsilon^f_{d2} + (\epsilon^f_l + \epsilon^f_u)\boldsymbol{u}^s(e) + \epsilon^f_{d1} \right) \right\} \\
&\le \epsilon^f_{d2} + |S_1|\cdot 3\epsilon^f_{d2} + |S_2|(2\epsilon^f_{d2} + \epsilon^f_{d1} + (\epsilon^f_l + \epsilon^f_u)\left\|\boldsymbol{u}^s\right\|_{max}) \\
&= (2|S_2| + 3|S_1| + 1)\epsilon^f_{d2} + |S_2|\epsilon^f_{d1} + |S_2|\left\|\boldsymbol{u}^s\right\|_{max}(\epsilon^f_l + \epsilon^f_u) \\
&\le 3(|S_1| + |S_2|)\left(\epsilon^f_{d1} + \epsilon^f_{d2} + \left\|\boldsymbol{u}^s\right\|_{max}(\epsilon^f_l + \epsilon^f_u) \right),
\end{aligned} \tag{61}$$

where we use $|S_2| \ge 1$ in the last inequality.

Having computed the error in demand for vertices other than s_i, t_i, now we compute the error in demand for the sources and sinks in G^s.

For source s_1, we notice that all outgoing flows of s_1 in G^f cannot route commodity 2 because there is no error in demand for s_1 in G^f by Eq. (51) in Definition 4.19. As we map back by setting $\boldsymbol{f}^s(e) = \boldsymbol{f}^f(e_1)$ for a selective edge e and map back trivially for non-selective edges, there is no error in demand for s_1, which validates Eq. (37) in Definition 4.16 about SFFA. Similarly, there is no error in demand for s_2.

For sinks t_1 and t_2, we compute the error in demand for t_1 first, and that of t_2 can be obtained directly by symmetry. We denote the error in demand for sink t_1 by $\bar{\tau}^s_{d2}$, and by Eq. (37) in

Definition 4.16, it can be computed as

$$\begin{aligned}
\bar{\tau}^s_{d2} &= \sum_{(x,t_1)\in E^s} \boldsymbol{f}^s_2(x,t_1) \\
&\overset{(1)}{=} \sum_{(x,t_1)\in S} \boldsymbol{f}^s_2(x,t_1) + \sum_{(x,t_1)\in E^s\setminus S} \boldsymbol{f}^s_2(x,t_1) \\
&\overset{(2)}{=} \sum_{e=(x,t_1)\in S} \boldsymbol{f}^f_2(e_1) + \sum_{(x,t_1)\in E^s\setminus S} \boldsymbol{f}^f_2(x,t_1) \\
&\overset{(3)}{\leq} \sum_{e=(x,t_1)\in S} \boldsymbol{f}^f_2(e_1) + \left(\epsilon^f_{d2} - \sum_{e=(x,t_1)\in S} \boldsymbol{f}^f_2(e_3) - \sum_{e\in S_1} \boldsymbol{f}^f_2(e_4)\right) \\
&\overset{(4)}{\leq} \epsilon^f_{d2} + \left|\sum_{e=(x,t_1)\in S} (\boldsymbol{f}^f_2(e_1) - \boldsymbol{f}^f_2(e_3))\right| \\
&\overset{(5)}{\leq} \tau^s_{d2}.
\end{aligned} \tag{62}$$

For step (1), we separate non-selective incoming edges from selective incoming edges of sink t_1. For step (2), we apply the rule of mapping $\boldsymbol{f}^f$ back to $\boldsymbol{f}^s$. For step (3), it comes from the error in demand of t_1 in G^f, where the incoming edges of t_1 include: e_4 of all selective edges for commodity 1; e_3 of all selective edges that incident to t_1; all non-selective edges that incident to t_1, i.e.,

$$\sum_{(x,t_1)\in E^f} \boldsymbol{f}^f_2(x,t_1) = \sum_{e\in S_1} \boldsymbol{f}^f_2(e_4) + \sum_{e=(x,t_1)\in S} \boldsymbol{f}^f_2(e_3) + \sum_{(x,t_1)\in E^s\setminus S} \boldsymbol{f}^f_2(x,t_1) \leq \epsilon^f_{d2}.$$

For step (4), we use $\sum_{e\in S_1} \boldsymbol{f}^f_2(e_4) \geq 0$. For step (5), we apply Eq. (61), where we have

$$\tau^s_{d2} = \epsilon^f_{d2} + \max_{y\in V^s\setminus\{s_2,t_2\}} \left|\sum_{e=(x,y)\in S} (\boldsymbol{f}^f_2(e_1) - \boldsymbol{f}^f_2(e_3))\right|.$$

Therefore, Eq. (62) indicates that error in demand of t_1 that can be achieved can be bounded by τ^s_{d2}. By symmetry, we have error in demand of t_2 that can be achieved can also be bounded τ^s_{d1}, i.e., $\bar{\tau}^s_{d1} \leq \tau^s_{d1}$. Thus, we can unify the error in demand for sinks and other non-fixed flow vertices.

3. Error in type.
   By Eq. (38) in Definition 4.16, error in type for edges selecting commodity i that we can achieve is computed as

$$\begin{aligned}
\tau^s_{ti} = \max_{e\in S_i} \boldsymbol{f}^s_{\bar{i}}(e) &= \max_{e\in S_i} \boldsymbol{f}^f_{\bar{i}}(e_1) \\
&\leq \max_{e\in S_i} (\boldsymbol{f}^f_{\bar{i}}(e_1) + \boldsymbol{f}^f_{\bar{i}}(e_2)) \quad \text{Becasue } \boldsymbol{f}^f_{\bar{i}}(e_2) \geq 0
\end{aligned}$$

Again, we first compute τ^s_{t1}, τ^s_{t2} can be obtained directly by symmetry. For an arbitrary edge $e = (x, y) \in S_1$, it is known that e_4 and e_5 are incident to t_1 and s_1. By error in demand on vertex xy, we have

$$\left|\boldsymbol{f}^f_2(e_1) + \boldsymbol{f}^f_2(e_2) - \boldsymbol{f}^f_2(e_4)\right| \leq \epsilon^f_{d2}.$$

As $\boldsymbol{f}_2^f(e_4) \le \epsilon_{d2}^f$ because of error in demand for t_1 in G^f, then we can bound

$$\boldsymbol{f}_2^f(e_1) + \boldsymbol{f}_2^f(e_2) \le 2\epsilon_{d2}^f.$$

Since e is an arbitrary edge in S_1, thus

$$\tau_{t1}^s \le 2\epsilon_{d2}^f. \tag{63}$$

By symmetry, we also have

$$\tau_{t2}^s \le 2\epsilon_{d1}^f. \tag{64}$$

Now, we prove that $\boldsymbol{f}^s$ solves the SFFA instance by showing the the error achieved by $\boldsymbol{f}^s$ is less than the error target set in the SFFA instance. Combining Eq. (52), (53), (60), (61), (63), (64), we have

$$\tau_u^s \le \epsilon_u^f, \qquad \tau_l^s \le \epsilon_l^f,$$

$$\tau_{d1}^s, \tau_{d2}^s \le 3(|S_1| + |S_2|)\left(\epsilon_{d1}^f + \epsilon_{d2}^f + \|\boldsymbol{u}^s\|_{max}\left(\epsilon_l^f + \epsilon_u^f\right)\right),$$

$$\tau_{t1}^s \le 2\epsilon_{d2}^f, \qquad \tau_{t2}^s \le 2\epsilon_{d1}^f.$$

As we set in the reduction that $\epsilon_l^f = \min\left\{\epsilon_l^s, \frac{\min\{\epsilon_{d1}^s, \epsilon_{d2}^s\}}{12(|S_1|+|S_2|)\|\boldsymbol{u}^s\|_{max}}\right\}$, $\epsilon_u^f = \min\left\{\epsilon_u^s, \frac{\min\{\epsilon_{d1}^s, \epsilon_{d2}^s\}}{12(|S_1|+|S_2|)\|\boldsymbol{u}^s\|_{max}}\right\}$, $\epsilon_{d1}^f = \min\left\{\frac{\min\{\epsilon_{d1}^s, \epsilon_{d2}^s\}}{12(|S_1|+|S_2|)}, \frac{\epsilon_{t2}^s}{2}\right\}$, $\epsilon_{d2}^f = \min\left\{\frac{\min\{\epsilon_{d1}^s, \epsilon_{d2}^s\}}{12(|S_1|+|S_2|)}, \frac{\epsilon_{t2}^s}{2}\right\}$, then we have

$$\tau_u^s \le \epsilon_u^f \le \epsilon_u^s, \qquad \tau_l^s \le \epsilon_l^f \le \epsilon_l^s,$$

$$\tau_{d1}^s \le 3(|S_1| + |S_2|)\left(\frac{\epsilon_{d1}^s}{12(|S_1| + |S_2|)} + \frac{\epsilon_{d1}^s}{12(|S_1| + |S_2|)} + \|\boldsymbol{u}^s\|_{max}\left(\frac{\epsilon_{d1}^s}{12(|S_1| + |S_2|)\|\boldsymbol{u}^s\|_{max}} + \frac{\epsilon_{d1}^s}{12(|S_1| + |S_2|)\|\boldsymbol{u}^s\|_{max}}\right)\right) \le \epsilon_{d1}^s,$$

$$\tau_{d2}^s \le 3(|S_1| + |S_2|)\left(\frac{\epsilon_{d2}^s}{12(|S_1| + |S_2|)} + \frac{\epsilon_{d2}^s}{12(|S_1| + |S_2|)} + \|\boldsymbol{u}^s\|_{max}\left(\frac{\epsilon_{d2}^s}{12(|S_1| + |S_2|)\|\boldsymbol{u}^s\|_{max}} + \frac{\epsilon_{d2}^s}{12(|S_1| + |S_2|)\|\boldsymbol{u}^s\|_{max}}\right)\right) \le \epsilon_{d2}^s,$$

$$\tau_{t1}^s \le \epsilon_{t1}^s, \qquad \tau_{t2}^s \le \epsilon_{t2}^s.$$

Thus, we can conclude that if $\boldsymbol{f}^f$ is a solution to the 2CFFA instance with the error $\epsilon_l^f, \epsilon_u^f, \epsilon_{d1}^f, \epsilon_{d2}^f$ as set in the reduction, we can map it back to a solution $\boldsymbol{f}^s$ to the SFFA instance with its error target achieved. □

4.8 2CFF(A) to 2CFR(A)

4.8.1 2CFF to 2CFR

We show the reduction from a 2CFF instance $(G^f, F^f, \boldsymbol{u}^f, s_1, t_1, s_2, t_2)$ to a 2CFR instance $(G^r, \boldsymbol{u}^r, \bar{s}_1, \bar{t}_1, \bar{s}_2, \bar{t}_2, R_1, R_2)$. First of all, we add two new sources $\bar{s}_1, \bar{s}_2$ and two new sinks $\bar{t}_1, \bar{t}_2$. Then, for each edge $e \in E^f$, we map it to a gadget consisting edges $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ in G^r,

as shown in the upper part of Figure 6. Additionally, there is another gadget with 5 edges in G^r that connects the original sink t_i and source s_i, as shown in the lower part of Figure 6. Capacity of these edges is the sum capacities of all edges in G^f, i.e., $M^f = \sum_{e \in E^f} \boldsymbol{u}^f(e)$. Additionally, we set $R_1 = R_2 = 2M^f$, indicating that at least $2M^f$ unit of flow should be routed from $\bar{s}_i$ to $\bar{t}_i$, $i \in \{1, 2\}$.

The key idea to remove the fixed flow constraint is utilizing edge directions and the requirements that $2M^f$ units of the flow of commodity i to be routed from the new source $\bar{s}_i$ to the new sink $\bar{t}_i$, $i \in \{1, 2\}$. It is noticed that all edges that are incident to the new sources and sinks should be saturated to fulfill the requirements. Therefore, for a fixed flow edge e in the first gadget, the incoming flow of vertex xy' and the outgoing flow of vertex xy must be $2u$. Since the capacity of e_2 is $2u - u = u$, then the flow of e_1, e_2, e_3 are forced to be u. As such, the fixed flow constraint can be simulated. Note that instead of simply copying non-fixed flow edges to G^r, we also need to map non-fixed flow edges to the designed gadget in this step. This guarantees that the requirement on flow values can be satisfied if the 2CFF instance is feasible.

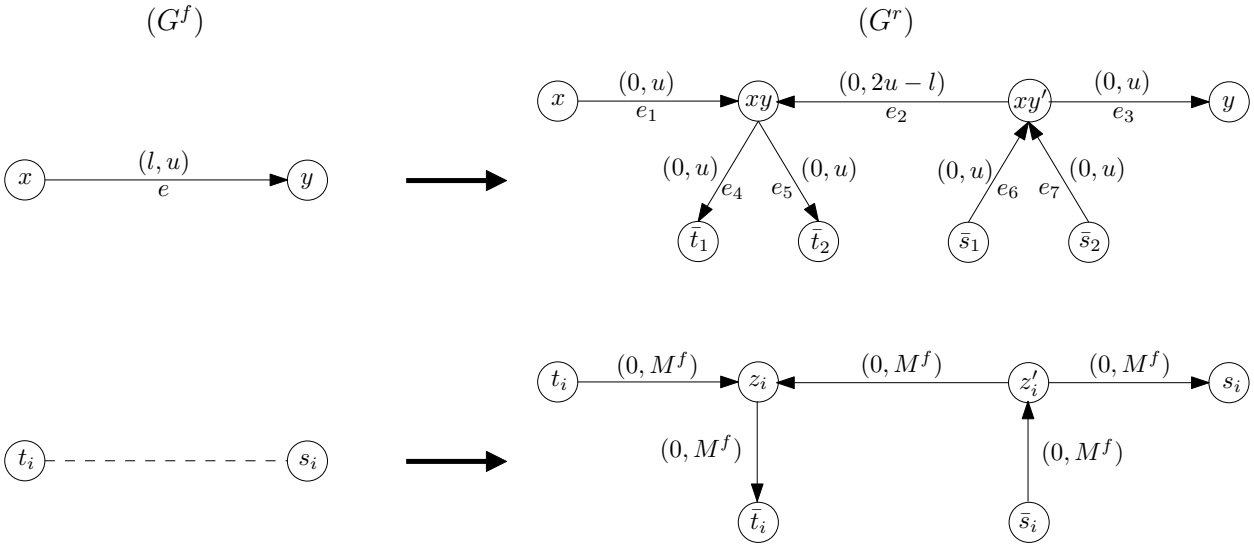

Figure 6: The reduction from 2CFF to 2CFR. $l = u$ if e is a fixed flow edge, and $l = 0$ if e is a non-fixed flow edge.

If a 2CFR solver returns $\boldsymbol{f}^r$ for the 2CFR instance $(G^r, \boldsymbol{u}^r, \bar{s}_1, \bar{t}_1, \bar{s}_2, \bar{t}_2, R_1, R_2)$, then we return $\boldsymbol{f}^f$ for the 2CFF instance $(G^f, F^f, \boldsymbol{u}^f, s_1, t_1, s_2, t_2)$ by setting $\boldsymbol{f}_i^f(e) = \boldsymbol{f}_i^r(e_1), \forall e \in E^f, i \in \{1, 2\}$. If the 2CFR solver returns "infeasible" for the 2CFR instance, then we return "infeasible" for the 2CFF instance.

Lemma 4.21 (2CFF to 2CFR)**.** *Given a* 2CFF *instance* $(G^f, F^f, \boldsymbol{u}^f, s_1, t_1, s_2, t_2)$*, we can construct, in time* $O(|E^f|)$*, a* 2CFR *instance* $(G^r, \boldsymbol{u}^r, \bar{s}_1, \bar{t}_1, \bar{s}_2, \bar{t}_2, R_1, R_2)$ *such that*

$$|V^r| = |V^f| + 2|E^f| + 8, \qquad |E^r| = 7|E^f| + 10, \qquad \|\boldsymbol{u}^r\|_{max} = \max\left\{2\left\|\boldsymbol{u}^f\right\|_{max}, M^f\right\},$$

$$R_1 = R_2 = 2M^f,$$

where $M^f = \sum_{e \in E^f} \boldsymbol{u}^f(e)$*. If the* 2CFF *instance has a solution, then the* 2CFR *instance has a solution.*

Proof. According to the reduction described above, from any solution $\boldsymbol{f}^f$ to the 2CFF instance, it is easy to derive a solution $\boldsymbol{f}^r$ to the 2CFR instance. Concretely, we define a feasible flow $\boldsymbol{f}^r$ as follows. For any edge $e \in E^f$, we set:

$$\boldsymbol{f}_i^r(e_1) = \boldsymbol{f}_i^r(e_3) = \boldsymbol{f}_i^f(e), \qquad i \in \{1,2\},$$
$$\boldsymbol{f}_i^r(e_2) = u - \boldsymbol{f}_i^f(e), \qquad i \in \{1,2\},$$
$$\boldsymbol{f}_1^r(e_4) = \boldsymbol{f}_1^r(e_6) = u, \qquad \boldsymbol{f}_2^r(e_5) = \boldsymbol{f}_2^r(e_7) = u,$$
$$\boldsymbol{f}_2^r(e_4) = \boldsymbol{f}_2^r(e_6) = \boldsymbol{f}_1^r(e_5) = \boldsymbol{f}_1^r(e_7) = 0,$$

where $u = \boldsymbol{u}^f(e)$. And we set

$$\boldsymbol{f}_i^r(t_i, z_i) = \boldsymbol{f}_i^r(z_i', s_i) = \sum_{w:(s_i,w)\in E^f} \boldsymbol{f}_i^f(s_i, w) \le M^f, \qquad i \in \{1,2\},$$

$$\boldsymbol{f}_i^r(z_i', z_i) = M^f - \sum_{w:(s_i,w)\in E^f} \boldsymbol{f}_i^f(s_i, w), \qquad i \in \{1,2\},$$

$$\boldsymbol{f}_i^r(\bar{s}_i, z_i') = \boldsymbol{f}_i^r(z_i, \bar{t}_i) = M^f, \qquad i \in \{1,2\},$$
$$\boldsymbol{f}_{\bar{i}}^r(t_i, z_i) = \boldsymbol{f}_{\bar{i}}^r(z_i', s_i) = \boldsymbol{f}_{\bar{i}}^r(z_i', z_i) = \boldsymbol{f}_{\bar{i}}^r(\bar{s}_i, z_i') = \boldsymbol{f}_{\bar{i}}^r(z_i, \bar{t}_i) = 0, \qquad \bar{i} \in \{1,2\}\backslash i.$$

Now, we prove that $\boldsymbol{f}^r$ is a solution to the 2CFR instance.

1. Capacity constraint.
   Only the capacity constraint on edge e_2 is nontrivial.

   - If e is a fixed flow edge, then we have $l = u$, thus the capacity of e_2 is u. By construction, we have
   $$\sum_{i=\{1,2\}} \boldsymbol{f}_i^r(e_2) = 2u - \sum_{i=\{1,2\}} \boldsymbol{f}_i^f(e) = 2u - u = u,$$
   where we use $\sum_{i=\{1,2\}} \boldsymbol{f}_i^f(e) = u$.
   - If e is a non-fixed flow edge, then we have $l = 0$, thus the capacity of e_2 is $2u$. By construction, we have
   $$\sum_{i=\{1,2\}} \boldsymbol{f}_i^r(e_2) = 2u - \sum_{i=\{1,2\}} \boldsymbol{f}_i^f(e) \le 2u,$$
   where we use $\sum_{i=\{1,2\}} \boldsymbol{f}_i^f(e) \ge 0$.

   Therefore, we conclude that $\boldsymbol{f}^r$ fulfills the capacity constraint.

2. Flow conservation constraint.
   Only the flow conservation constraint on vertices xy, xy', z_i, z_i' are nontrivial.

   - For vertex xy, we have
   $$\boldsymbol{f}_1^r(e_1) + \boldsymbol{f}_1^r(e_2) = \boldsymbol{f}_1^f(e) + (u - \boldsymbol{f}_1^f(e)) = u = \boldsymbol{f}_1^r(e_4),$$
   and
   $$\boldsymbol{f}_2^r(e_1) + \boldsymbol{f}_2^r(e_2) = \boldsymbol{f}_2^f(e) + (u - \boldsymbol{f}_2^f(e)) = u = \boldsymbol{f}_2^r(e_5).$$
   We can prove for vertex xy' similarly.

- For vertex $z_i, i \in \{1,2\}$, we have

$$\begin{aligned}\boldsymbol{f}_i^r(t_i, z_i) + \boldsymbol{f}_i^r(z_i', z_i) &= \sum_{w:(s_i,w)\in E^f} \boldsymbol{f}_i^f(s_i, w) + \left(M^f - \sum_{w:(s_i,w)\in E^f} \boldsymbol{f}_i^f(s_i, w)\right) \\ &= M^f = \boldsymbol{f}_i^r(z_i, \bar{t}_i).\end{aligned}$$

  We can prove for vertex z_i' similarly.

Therefore, we conclude that $\boldsymbol{f}^r$ fulfills the flow conservation constraint.

3. Requirement $R_1 = R_2 = 2M^f$.
   For commodity 1, we have

$$R_1 = \sum_{w:(\bar{s}_1,w)\in E^r} \boldsymbol{f}_1^r(\bar{s}_1, w) = \boldsymbol{f}^r(\bar{s}_1, z_1') + \sum_{e\in E^f} \boldsymbol{f}_1^r(e_6) = M^f + \sum_{e\in E^f} \boldsymbol{u}^f(e) = 2M^f,$$

$$R_1 = \sum_{u:(u,\bar{t}_1)\in E^r} \boldsymbol{f}_1^r(u, \bar{t}_1) = \boldsymbol{f}^r(z_1, \bar{t}_1) + \sum_{e\in E^f} \boldsymbol{f}_1^r(e_4) = M^f + \sum_{e\in E^f} \boldsymbol{u}^f(e) = 2M^f.$$

   We can prove that the requirement $R_2 = 2M^f$ is satisfied similarly.

To conclude, $\boldsymbol{f}^r$ is a feasible flow to the 2CFR instance.

Now, we track the change of problem size after reduction. Based on the reduction method, given a 2CFF instance with $|V^f|$ vertices and $|E^f|$ edges (including $|F^f|$ fixed flow edges), we can compute the size of the reduced 2CFR instance as follows.

1. $|V^r|$ vertices. First, all vertices in V^f are maintained. Then, for each edge $e = (x, y)$, two new auxiliary vertices xy, xy' are added. And finally, in the second gadget, $\{\bar{s}_i, \bar{t}_i, z_i, z_i'\}, i \in \{1,2\}$ are added. Hence, we have
$$|V^r| = |V^f| + 2|E^f| + 8.$$

2. $|E^r|$ edges. First, each edge $e \in E^f$ is replaced by the first gadget in Figure 6 with 7 edges $(e_1, \ldots, e_7)$. Then, 5 edges are added between t_i and s_i according to the second gadget in Figure 6, $i \in \{1,2\}$. Thus, we have
$$|E^r| = 7|E^f| + 10.$$

3. The maximum edge capacity is bounded by
$$\|\boldsymbol{u}^r\|_{max} = \max\left\{2\left\|\boldsymbol{u}^f\right\|_{max}, M^f\right\}.$$

   For all fixed flow edges $e \in F^f$ with fixed flow $\boldsymbol{u}^f(e)$, the 7 edges $(e_1, \ldots, e_7)$ in the first gadget are all with capacity $\boldsymbol{u}^f(e)$. For all non-fixed flow edges in $e \in E^f \backslash F^f$ with capacity $\boldsymbol{u}^f(e)$, the capacity of e_2 becomes $2\boldsymbol{u}^f(e)$ and the capacity of the rest 6 edges in the first gadget remains $\boldsymbol{u}^f(e)$. Moreover, the 5 edges in the second gadget are all with capacity M^f, which is the sum of the capacity of all edges.

To estimate the reduction time, it is observed that it takes constant time to reduce each edge in E^f since only a constant number of vertices and edges are added. Thus, the reduction of this step takes $O(|E^f|)$ time. □

4.8.2 2CFFA to 2CFRA

The above lemma shows the reduction between exactly solving a 2CFF instance and exactly solving a 2CFR instance. Next, we generalize the case with exact solutions to the case that allows approximate solutions. First of all, we give a definition of the approximate version of 2CFR.

Definition 4.22 (2CFR Approximate Problem (2CFRA))**.** A 2CFRA instance is given by a 2CFR instance $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2, R_1, R_2)$ as in Definition 2.8, and error parameters $\epsilon, \epsilon_1, \epsilon_2 \in [0, 1]$, which we collect in a tuple $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2, R_1, R_2, \epsilon, \epsilon_1, \epsilon_2)$. We say an algorithm solves the 2CFRA problem, if, given any 2CFRA instance, it returns a pair of flows $\boldsymbol{f}_1, \boldsymbol{f}_2 \geq \boldsymbol{0}$ that satisfies

$$0 \leq \boldsymbol{f}_1(e) + \boldsymbol{f}_2(e) \leq (1+\epsilon)\boldsymbol{u}(e), \quad \forall e \in E \tag{65}$$

$$F_i \geq R_i - \epsilon_i, \quad i \in \{1, 2\} \tag{66}$$

or it correctly declares that the associated 2CFR instance is infeasible. We refer to the error in (65) as error in congestion, and the error in (66) as error in requirement.

We can use the same reduction method and solution mapping method in the exact case to the approximate case.

Lemma 4.23 (2CFFA to 2CFRA)**.** *Given a* 2CFFA *instance* $\left(G^f, F^f, \boldsymbol{u}^f, s_1, t_1, s_2, t_2, \epsilon_l^f, \epsilon_u^f, \epsilon_{d1}^f, \epsilon_{d2}^f\right)$*, we can construct, in* $O(|E^f|)$ *time, a* 2CFRA *instance* $(G^r, \boldsymbol{u}^r, \bar{s}_1, \bar{t}_1, \bar{s}_2, \bar{t}_2, R_1, R_2, \epsilon^r, \epsilon_1^r, \epsilon_2^r)$ *such that*

$$|V^r| = |V^f| + 2|E^f| + 8, \qquad |E^r| = 7|E^f| + 10, \qquad \|\boldsymbol{u}^r\|_{max} = \max\left\{2\left\|\boldsymbol{u}^f\right\|_{max}, M^f\right\},$$

$$R_1 = R_2 = 2M^f,$$

and

$$\epsilon^r = \min\left\{\frac{\epsilon_l^f}{12M^f},\ \epsilon_u^f,\ \frac{\epsilon_{d1}^f}{8M^f},\ \frac{\epsilon_{d2}^f}{8M^f}\right\}, \qquad \epsilon_1^r = \min\left\{\frac{\epsilon_l^f}{3},\ \frac{\epsilon_{d1}^f}{4}\right\}, \qquad \epsilon_2^r = \min\left\{\frac{\epsilon_l^f}{3},\ \frac{\epsilon_{d2}^f}{4}\right\},$$

where $M^f = \sum_{e \in E^f} \boldsymbol{u}^f(e)$*. If the* 2CFF *instance* $\left(G^f, F^f, \boldsymbol{u}^f, s_1, t_1, s_2, t_2\right)$ *has a solution, then the* 2CFRA *instance* $(G^r, \boldsymbol{u}^r, \bar{s}_1, \bar{t}_1, \bar{s}_2, \bar{t}_2, R_1, R_2)$ *has a solution. Furthermore, if* $\boldsymbol{f}^r$ *is a solution to the* 2CFRA *instance, then in time* $O(|E^f|)$*, we can compute a solution* $\boldsymbol{f}^f$ *to the* 2CFFA *instance.*

Proof. Since we use the same reduction method in the exact case to the approximate case, the conclusions in Lemma 4.21 also apply here, including the reduction time, problem size, and that 2CFR has a feasible solution when 2CFF has one. It remains to show the solution mapping time, as well as how the problem error changes by mapping an approximate solution to 2CFRA back to an approximate solution to 2CFFA.

Based on the solution mapping method described above, it takes constant time to set the value of each of $\boldsymbol{f}^f$, and $\boldsymbol{f}^f$ has $|E^f|$ entries, such a solution mapping takes $O(|E^f|)$ time.

Now, we conduct an error analysis. The mapping will generate two types of error: (1) error in congestion; (2) error in demand.

1. Error in congestion.
   We first track the error of the upper bound of capacity. For any edge $e \in E^f$, the maximum flow to be routed in e is

$$(1+\tau_u^f)\boldsymbol{u}^f(e) = \sum_{i=\{1,2\}} \boldsymbol{f}_i^f(e) = \sum_{i=\{1,2\}} \boldsymbol{f}_i^r(e_1) \leq (1+\epsilon^r)\boldsymbol{u}^r(e_1) = (1+\epsilon^r)\boldsymbol{u}^f(e),$$

thus,

$$\tau_u^f \le \epsilon^r.$$

As we set $\epsilon^r \le \epsilon_u^f$ in the reduction, it indicates that $\tau_u^f \le \epsilon_u^f$.

Next, we track the error of the lower bound of capacity. Note that we only need to take fixed flow edges into account since the lower capacity bound for non-fixed flow edges is 0. By error in congestion as shown in Eq. (65) and error in requirement as shown in Eq. (66), as well as $R_1 = R_2 = 2M^f$, we have

$$2M^f - \epsilon_1^r \le \sum_{e \in E^f} \boldsymbol{f}_1^r(e_4) + \boldsymbol{f}^r(z_1, \bar{t}_1) \le 2(1+\epsilon^r)M^f,$$

$$2M^f - \epsilon_2^r \le \sum_{e \in E^f} \boldsymbol{f}_2^r(e_5) + \boldsymbol{f}^r(z_2, \bar{t}_2) \le 2(1+\epsilon^r)M^f,$$

We now try to compute the minimum flow that can be routed through any fixed flow edge in F^f. For an arbitrary $\hat{e} \in F^f$, we can rearrange the above equations and have

$$\begin{aligned}
\boldsymbol{f}_1^r(\hat{e}_4) &\ge 2M^f - \epsilon_1^r - \sum_{e \in E^f \setminus \hat{e}} \boldsymbol{f}_1^r(e_4) - \boldsymbol{f}^r(z_1, \bar{t}_1) \\
&\ge \max\left\{0, 2M^f - \epsilon_1^r - \sum_{e \in E^f \setminus \hat{e}} (1+\epsilon^r)\boldsymbol{u}^f(e) - (1+\epsilon^r)M^f\right\} && \epsilon^r \text{ error in congestion in } G^r \\
&= \max\left\{0, 2M^f - \epsilon_1^r - (1+\epsilon^r)(2M^f - \boldsymbol{u}^f(\hat{e}))\right\} && \text{Because } M^f = \sum_{e \in E^f} \boldsymbol{u}^f(e) \\
&= \max\left\{0, (1+\epsilon^r)\boldsymbol{u}^f(\hat{e}) - 2M^f\epsilon^r - \epsilon_1^r\right\}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\boldsymbol{f}_2^r(\hat{e}_5) &\ge 2M^f - \epsilon_2^r - \sum_{e \in E^f \setminus \hat{e}} \boldsymbol{f}_1^r(e_5) - \boldsymbol{f}^r(z_2, \bar{t}_2) \\
&\ge \max\left\{0, (1+\epsilon^r)\boldsymbol{u}^f(\hat{e}) - 2M^f\epsilon^r - \epsilon_2^r\right\}.
\end{aligned}$$

Therefore, for any fixed flow edge $\hat{e} = (x, y) \in F^f$, the minimum flow to be routed can be lower bounded by

$$\begin{aligned}
(1-\tau_l^f)\boldsymbol{u}^f(\hat{e}) = \boldsymbol{f}^f(\hat{e}) &= \boldsymbol{f}^r(\hat{e}_1) \\
&\overset{(1)}{=} \boldsymbol{f}_1^r(\hat{e}_4) + \boldsymbol{f}_2^r(\hat{e}_5) - \boldsymbol{f}^r(\hat{e}_2) \\
&\overset{(2)}{\ge} \max\{0, 2(1+\epsilon^r)\boldsymbol{u}^f(\hat{e}) - 4M^f\epsilon^r - \epsilon_1^r - \epsilon_2^r - (1+\epsilon^r)\boldsymbol{u}^f(\hat{e})\} \\
&= \max\{0, (1+\epsilon^r)\boldsymbol{u}^f(\hat{e}) - 4M^f\epsilon^r - \epsilon_1^r - \epsilon_2^r\} \\
&= \max\left\{0, \left(1 + \epsilon^r - \frac{4M^f\epsilon^r + \epsilon_1^r + \epsilon_2^r}{\boldsymbol{u}^f(\hat{e})}\right)\boldsymbol{u}^f(\hat{e})\right\} \\
&= \max\left\{0, \left(1 - \left(\left(\frac{4M^f}{\boldsymbol{u}^f(\hat{e})} - 1\right)\epsilon^r + \frac{\epsilon_1^r + \epsilon_2^r}{\boldsymbol{u}^f(\hat{e})}\right)\right)\boldsymbol{u}^f(\hat{e})\right\}.
\end{aligned} \tag{67}$$

For step (1), we apply the flow conservation constraint on vertex xy since there is no error in demand in G^r. For step (2), we plug in the lower bound of $\boldsymbol{f}_1^r(\hat{e}_4), \boldsymbol{f}_2^r(\hat{e}_5)$ that are proved previously, and the upper bound of $\boldsymbol{f}_1^r(\hat{e}_2)$. Thus, we have

$$\tau_l^f \leq \left(\frac{4M^f}{\boldsymbol{u}^f(\hat{e})} - 1\right)\epsilon^r + \frac{\epsilon_1^r + \epsilon_2^r}{\boldsymbol{u}^f(\hat{e})} \leq 4M^f\epsilon^r + \epsilon_1^r + \epsilon_2^r,$$

where we use $\boldsymbol{u}^f(\hat{e}) \geq 1$ because of integer capacities.

As we set in the reduction that

$$\epsilon^r \leq \frac{\epsilon_l^f}{12M^f}, \quad \epsilon_1^r \leq \frac{\epsilon_l^f}{3}, \quad \epsilon_2^r \leq \frac{\epsilon_l^f}{3},$$

thus we have

$$\tau_l^f \leq 4M^f\frac{\epsilon_l^f}{12M^f} + \frac{\epsilon_l^f}{3} + \frac{\epsilon_l^f}{3} \leq \epsilon_l^f.$$

2. Error in demand.
We focus on the error in demand for commodity 1 first, and then the error in demand for commodity 2 can be achieved directly by symmetry. By Eq. (50) in Definition 4.19, error in demand of commodity 1 that we can achieve for vertices other than s_1, t_1 in G^f is computed as

$$\begin{aligned}
\tau_{d1}^f &= \max_{y\in V^f\setminus\{s_1,t_1\}}\left|\sum_{(x,y)\in E^f}\boldsymbol{f}_1^f(x,y) - \sum_{(y,z)\in E^f}\boldsymbol{f}_1^f(y,z)\right| \\
&\overset{(1)}{=} \max_{y\in V^f\setminus\{s_1,t_1\}}\left|\sum_{e=(x,y)\in E^f}\boldsymbol{f}_1^r(e_1) - \sum_{e=(y,z)\in E^f}\boldsymbol{f}_1^r(e_1)\right| \\
&\overset{(2)}{=} \max_{y\in V^f\setminus\{s_1,t_1\}}\left|\sum_{e=(x,y)\in E^f}\boldsymbol{f}_1^r(e_1) - \sum_{e=(x,y)\in E^f}\boldsymbol{f}_1^r(e_3)\right| \\
&= \max_{y\in V^f\setminus\{s_1,t_1\}}\left|\sum_{e=(x,y)\in E^f}(\boldsymbol{f}_1^r(e_1) - \boldsymbol{f}_1^r(e_3))\right| \\
&\leq \max_{y\in V^f\setminus\{s_1,t_1\}}\sum_{e=(x,y)\in E^f}|\boldsymbol{f}_1^r(e_1) - \boldsymbol{f}_1^r(e_3)|.
\end{aligned} \tag{68}$$

For step (1), we apply the rule of mapping $\boldsymbol{f}^r$ back to $\boldsymbol{f}^f$. For step (2), we replace the sum of outgoing flows of y by the sum of its incoming flows without introducing any error because there is no error in demand in G^r, i.e.,

$$\sum_{e=(y,z)\in E^f}\boldsymbol{f}_1^r(e_1) = \sum_{e=(x,y)\in E^f}\boldsymbol{f}_1^r(e_3).$$

Now, we try to bound $|\boldsymbol{f}_1^r(e_1) - \boldsymbol{f}_1^r(e_3)|$. By the flow conservation constraint of vertex xy and xy' with respect to commodity 1, we have

$$\boldsymbol{f}_1^r(e_1) + \boldsymbol{f}_1^r(e_2) = \boldsymbol{f}_1^r(e_4),$$

$$\boldsymbol{f}_1^r(e_3) + \boldsymbol{f}_1^r(e_2) = \boldsymbol{f}_1^r(e_6).$$

Subtracting these two equations gives

$$|\boldsymbol{f}_1^r(e_1) - \boldsymbol{f}_1^r(e_3)| = |\boldsymbol{f}_1^r(e_4) - \boldsymbol{f}_1^r(e_6)|\,,$$

which can be plugged back into Eq. (68) to give

$$\tau_{d1}^f \leq \max_{y \in V^f \setminus \{s_1, t_1\}} \sum_{e=(x,y) \in E^f} |\boldsymbol{f}_1^r(e_4) - \boldsymbol{f}_1^r(e_6)|\,.$$

The following is a claim that shows the sum of $|\boldsymbol{f}_1^r(e_4) - \boldsymbol{f}_1^r(e_6)|$ for all edges $e \in E^f$ can be upper bounded.

Claim 4.24.

$$\sum_{e \in E^f} |\boldsymbol{f}_1^r(e_4) - \boldsymbol{f}_1^r(e_6)| \leq 4M^f \epsilon^r + 2\epsilon_1^r.$$

By Claim 4.24, we have

$$\begin{aligned}\tau_{d1}^f &\leq \max_{y \in V^f \setminus \{s_1, t_1\}} \sum_{e=(x,y) \in E^f} |\boldsymbol{f}_1^r(e_4) - \boldsymbol{f}_1^r(e_6)| \\ &\leq \sum_{e \in E^f} |\boldsymbol{f}_1^r(e_4) - \boldsymbol{f}_1^r(e_6)| \leq 4M^f \epsilon^r + 2\epsilon_1^r.\end{aligned}$$

As we set in the reduction $\epsilon^r \leq \frac{\epsilon_{d1}^f}{8M^f}, \epsilon_1^r \leq \frac{\epsilon_{d1}^f}{4}$, it indicates that

$$\tau_{d1}^f \leq 4M^f \frac{\epsilon_{d1}^f}{8M^f} + 2\frac{\epsilon_{d1}^f}{4} = \epsilon_{d1}^f. \tag{69}$$

By symmetry of the two commodities, we can also bound the error in demand for commodity 2 by

$$\tau_{d2}^f \leq 4M^f \epsilon^r + 2\epsilon_2^r.$$

As we set in the reduction $\epsilon^r \leq \frac{\epsilon_{d2}^f}{8M^f}, \epsilon_2^r \leq \frac{\epsilon_{d2}^f}{4}$, it indicates that

$$\tau_{d2}^f \leq 4M^f \frac{\epsilon_{d2}^f}{8M^f} + 2\frac{\epsilon_{d2}^f}{4} = \epsilon_{d2}^f. \tag{70}$$

Having computed the error in demand for vertices other than s_i, t_i, now we compute the error in demand for sources and sinks in G^f.

For source s_1, it is noticed that all outgoing flows of s_1 in G^r cannot route commodity 2 because the incoming flow of s_1 is only from z_1' and thus from $\bar{s}_1$, where only flow of commodity 1 is allowed since there is no error in demand in G^r. As we map back by setting $\boldsymbol{f}^f(e) = \boldsymbol{f}^r(e_1)$ for a fixed flow edge e and map back trivially for non-fixed flow edges, there is no error in demand for s_1, which validates Eq. (51) in Definition 4.19 about 2CFFA. Similarly, there is no error in demand for s_2.

For sinks t_1 and t_2, we compute the error in demand for t_1 first, and that of t_2 can be obtained directly by symmetry. We denote the error in demand for sink t_1 by $\bar{\tau}^f_{d2}$, and again by Eq. (51) in Definition 4.19, it can be computed as

$$\begin{aligned}\bar{\tau}^f_{d2} &= \sum_{(x,t_1)\in E^f} \boldsymbol{f}^f_2(x,t_1) \\ &\overset{(1)}{=} \sum_{e=(x,t_1)\in E^f} \boldsymbol{f}^r_2(e_1) \\ &\overset{(2)}{=} \left| \sum_{e=(x,t_1)\in E^f} (\boldsymbol{f}^r_2(e_1) - \boldsymbol{f}^r_2(e_3)) \right| \\ &\overset{(3)}{\leq} \tau^f_{d2}.\end{aligned} \tag{71}$$

For step (1), we apply the rule of mapping $\boldsymbol{f}^r$ back to $\boldsymbol{f}^f$. For step (2), we utilize the property that all incoming flows of t_1 in G^r cannot route commodity 2, i.e., $\boldsymbol{f}^r_2(e_3) = 0$, because the outgoing flow of t_1 is only to z_1 and then $\bar{t}_1$, where only flow of commodity 1 is allowed. For step (3), we apply the conclusion of Eq. (68) by symmetry that $\tau^f_{d2} = \max_{y\in V^f\setminus\{s_2,t_2\}} \left|\sum_{e=(x,y)\in E^f} (\boldsymbol{f}^r_2(e_1) - \boldsymbol{f}^r_2(e_3))\right|$. Therefore, Eq. (71) indicates that error in demand of t_1 that can be achieved can be bounded by τ^f_{d2}.

By symmetry, we have error in demand of t_2 that can be achieved is upper bounded by τ^f_{d1}, i.e., $\bar{\tau}^f_{d1} \leq \tau^f_{d1}$. Thus, we can unify the error in demand for sinks and other non-fixed flow vertices. And Eq. (69) and (70) also implies

$$\bar{\tau}^f_{d1} \leq \epsilon^f_{d1}, \qquad \bar{\tau}^f_{d2} \leq \epsilon^f_{d2}.$$

To summarize, if $\boldsymbol{f}^r$ is a solution to the 2CFRA instance with the error ϵ^r as set in the reduction, we can map it back to a solution $\boldsymbol{f}^f$ to the 2CFFA instance with its error target achieved.

□

Now, we provide a proof to Claim 4.24.

Proof of Claim 4.24. We divide all edges in E^f into two groups:

$$E_I = \{e \in E^f \ s.t.\ \boldsymbol{f}^r_1(e_4) \leq \boldsymbol{f}^r_1(e_6)\},$$

$$E_{II} = \{e \in E^f \ s.t.\ \boldsymbol{f}^r_1(e_4) > \boldsymbol{f}^r_1(e_6)\}.$$

We denote

$$T_I = \sum_{e\in E_I} \boldsymbol{f}^r_1(e_4), \qquad S_I = \sum_{e\in E_I} \boldsymbol{f}^r_1(e_6);$$

$$T_{II} = \sum_{e\in E_{II}} \boldsymbol{f}^r_1(e_4), \qquad S_{II} = \sum_{e\in E_{II}} \boldsymbol{f}^r_1(e_6).$$

Then,

$$\sum_{e\in E^f} |\boldsymbol{f}^r_1(e_4) - \boldsymbol{f}^r_1(e_6)| = (S_I - T_I) + (T_{II} - S_{II})$$

On one hand, by the capacity constraints, we have

$$S_I + T_{II} \leq (1+\epsilon^r) \sum_{e \in E^f} \boldsymbol{u}^f(e) = (1+\epsilon^r) M^f. \tag{72}$$

On the other hand, applying error in requirement, we have

$$T_I + T_{II} \geq (2M^f - \epsilon_1^r) - \boldsymbol{f}_1^r(z_1, \bar{t}_1) \geq 2M^f - \epsilon_1^r - (1+\epsilon^r)M^f = (1-\epsilon^r)M^f - \epsilon_1^r,$$
$$S_I + S_{II} = (2M^f - \epsilon_1^r) - \boldsymbol{f}_1^r(\bar{s}_1, z_1') \geq 2M^f - \epsilon_1^r - (1+\epsilon^r)M^f = (1-\epsilon^r)M^f - \epsilon_1^r.$$

Thus,

$$S_I + T_{II} = S_I + (T_I + T_{II}) - T_I \geq (1-\epsilon^r)M^f - \epsilon_1^r + (S_I - T_I),$$
$$S_I + T_{II} = (S_I + S_{II}) + T_{II} - S_{II} \geq (1-\epsilon^r)M^f - \epsilon_1^r + (T_{II} - S_{II}).$$

Together with Eq. (72), we obtain

$$S_I - T_I \leq S_I + T_{II} + \epsilon_1^r - (1-\epsilon^r)M^f \leq (1+\epsilon^r)M^f + \epsilon_1^r - (1-\epsilon^r)M^f = 2M^f\epsilon^r + \epsilon_1^r,$$
$$T_{II} - S_{II} \leq S_I + T_{II} + \epsilon_1^r - (1-\epsilon^r)M^f \leq (1+\epsilon^r)M^f + \epsilon_1^r - (1-\epsilon^r)M^f = 2M^f\epsilon^r + \epsilon_1^r.$$

Hence, we have

$$\sum_{e \in E^f} |\boldsymbol{f}_1^r(e_4) - \boldsymbol{f}_1^r(e_6)| = (S_I - T_I) + (T_{II} - S_{II}) \leq 4M^f\epsilon^r + 2\epsilon_1^r,$$

which finishes the proof. □

4.9 2CFR(A) to 2CF(A)

4.9.1 2CFR to 2CF

We show the reduction from a 2CFR instance $(G^r, \boldsymbol{u}^r, \bar{s}_1, \bar{t}_1, \bar{s}_2, \bar{t}_2, R_1, R_2)$ to a 2CF instance $(G^{2cf}, \boldsymbol{u}^{2cf}, \bar{\bar{s}}_1, \bar{t}_1, \bar{\bar{s}}_2, \bar{t}_2, R^{2cf})$. We copy the entire graph structure of G^r to G^{2cf}. In addition, we add to G^{2cf} with two new sources $\bar{\bar{s}}_1, \bar{\bar{s}}_2$ and two new edges $(\bar{\bar{s}}_1, \bar{s}_1), (\bar{\bar{s}}_2, \bar{s}_2)$ with capacity R_1, R_2, respectively. We set $R^{2cf} = R_1 + R_2$.

If a 2CF solver returns $\boldsymbol{f}^{2cf}$ for the 2CF instance $(G^{2cf}, \boldsymbol{u}^{2cf}, \bar{\bar{s}}_1, \bar{t}_1, \bar{\bar{s}}_2, \bar{t}_2, R^{2cf})$, then we return $\boldsymbol{f}^r$ for the 2CFR instance $(G^r, \boldsymbol{u}^r, \bar{s}_1, \bar{t}_1, \bar{s}_2, \bar{t}_2, R_1, R_2)$ by setting $\boldsymbol{f}_i^r(e) = \boldsymbol{f}_i^{2cf}(e), \forall e \in E^r, i \in \{1,2\}$. If the 2CF solver returns "infeasible" for the 2CF instance, then we return "infeasible" for the 2CFR instance.

Lemma 4.25 (2CFR to 2CF)**.** *Given a* 2CFR *instance* $(G^r, \boldsymbol{u}^r, \bar{s}_1, \bar{t}_1, \bar{s}_2, \bar{t}_2, R_1, R_2)$*, we can construct, in time* $O(|E^r|)$*, a* 2CF *instance* $(G^{2cf}, \boldsymbol{u}^{2cf}, \bar{\bar{s}}_1, \bar{t}_1, \bar{\bar{s}}_2, \bar{t}_2, R^{2cf})$ *such that*

$$|V^{2cf}| = |V^r| + 2, \qquad |E^{2cf}| = |E^r| + 2, \qquad \left\|\boldsymbol{u}^{2cf}\right\|_{max} = \max\{R_1, R_2\}, \qquad R^{2cf} = R_1 + R_2,$$

and if the 2CFR *instance has a solution, then the* 2CF *instance has a solution.*

Proof. According to the reduction described above, if there exists a solution $\boldsymbol{f}^r$ to the 2CFR instance such that $F_1^r \geq R_1, F_2^r \geq R_2$, then there also exists a solution $\boldsymbol{f}^{2cf}$ to the 2CF instance because $F_1^{2cf} = R_1, F_2^{2cf} = R_2$, and thus $F_1^{2cf} + F_2^{2cf} = R^{2cf}$.

For the problem size after reduction, it is obvious that

$$|V^{2cf}| = |V^r| + 2, \qquad |E^{2cf}| = |E^r| + 2, \qquad \left\|\boldsymbol{u}^{2cf}\right\|_{max} = \max\{R_1, R_2\}.$$

For the reduction time, it takes constant time to add two new vertices and edges, and it take $O(|E^r|)$ time to copy the rest edges. Thus, the reduction of this step can be performed in $O(|E^r|)$ time. □

4.9.2 2CFRA to 2CFA

The above lemma shows the reduction between exactly solving a 2CFR instance and exactly solving a 2CF instance. Next, we generalize the case with exact solutions to the case that allows approximate solutions.

To start with, we use the same reduction method and solution mapping method in the exact case to the approximate case.

Lemma 4.26 (2CFRA to 2CFA)**.** *Given a* 2CFRA *instance* $(G^r, \boldsymbol{u}^r, \bar{s}_1, \bar{t}_1, \bar{s}_2, \bar{t}_2, R_1, R_2, \epsilon^r, \epsilon_1^r, \epsilon_2^r)$*, we can construct, in time* $O(|E^r|)$*, a* 2CFA *instance* $(G^{2cf}, \boldsymbol{u}^{2cf}, \bar{\bar{s}}_1, \bar{t}_1, \bar{\bar{s}}_2, \bar{t}_2, R^{2cf}, \epsilon^{2cf})$ *such that*

$$|V^{2cf}| = |V^r| + 2, \qquad |E^{2cf}| = |E^r| + 2, \qquad \left\|\boldsymbol{u}^{2cf}\right\|_{max} = \max\{R_1, R_2\}, \qquad R^{2cf} = R_1 + R_2,$$

and

$$\epsilon^{2cf} = \min\left\{\epsilon^r, \frac{\epsilon_1^r}{R_2}, \frac{\epsilon_2^r}{R_1}\right\}.$$

If the 2CFR *instance* $(G^r, \boldsymbol{u}^r, \bar{s}_1, \bar{t}_1, \bar{s}_2, \bar{t}_2, R_1, R_2)$ *has a solution, then the* 2CF *instance* $(G^{2cf}, \boldsymbol{u}^{2cf}, \bar{\bar{s}}_1, \bar{t}_1, \bar{\bar{s}}_2, \bar{t}_2, R^{2cf})$ *has a solution. Furthermore, if* $\boldsymbol{f}^{2cf}$ *is a solution to the* 2CFA *instance, then in time* $O(|E^r|)$*, we can compute a solution* $\boldsymbol{f}^r$ *to the* 2CFRA *instance.*

Proof. Since we use the same reduction method in the exact case to the approximate case, the conclusions in Lemma 4.25 also apply here, including the reduction time, problem size, and that 2CF has a feasible solution when 2CFR has one. It remains to show the solution mapping time, as well as how the problem error changes by mapping an approximate solution to 2CFA back to an approximate solution to 2CFRA.

Based on the solution mapping method described above, it takes constant time to set the value of each of $\boldsymbol{f}^r$, and $\boldsymbol{f}^r$ has $|E^r|$ entries, such a solution mapping takes $O(|E^r|)$ time.

Now, we conduct an error analysis. If $\boldsymbol{f}^{2cf}$ is a solution to the 2CFA instance $(G^{2cf}, \boldsymbol{u}^{2cf}, \bar{\bar{s}}_1, \bar{t}_1, \bar{\bar{s}}_2, \bar{t}_2, R^{2cf}, \epsilon^{2cf})$, then by Definition 2.6 and Lemma 2.7, we have

$$F_1^{2cf} + F_2^{2cf} \geq R^{2cf}. \tag{73}$$

By error in congestion, we have

$$F_i^{2cf} = \boldsymbol{f}_1^{2cf}(\bar{\bar{s}}_i, \bar{s}_i) \leq (1 + \epsilon^{2cf})R_i, \qquad i \in \{1, 2\}. \tag{74}$$

Combining Eq. (73) and (74), we have

$$\begin{aligned} R_1 - R_2\epsilon^{2cf} \leq F_1^{2cf} \leq (1 + \epsilon^{2cf})R_1, \\ R_2 - R_1\epsilon^{2cf} \leq F_2^{2cf} \leq (1 + \epsilon^{2cf})R_2. \end{aligned} \tag{75}$$

As we map $\boldsymbol{f}^{2cf}$ back to $\boldsymbol{f}^r$ by setting

$$\boldsymbol{f}_i^r(e) = \boldsymbol{f}_i^{2cf}(e), \qquad \forall e \in E^r, i \in \{1, 2\},$$

we have the error in congestion that can be achieved by $\boldsymbol{f}^r$ is

$$\tau^r \leq \epsilon^{2cf}.$$

As we set in the reduction that $\epsilon^{2cf} \leq \epsilon^r$, then we have

$$\tau^r \leq \epsilon^r.$$

In addition, by the flow conservation, we have

$$F_i^r = \sum_{(\bar{s}_i,v)\in E^r} \boldsymbol{f}_i^r(\bar{s}_i,v) = \sum_{(\bar{s}_i,v)\in E^{2cf}} \boldsymbol{f}_i^{2cf}(\bar{s}_i,v) = \boldsymbol{f}_1^{2cf}(\bar{s}_i,\bar{s}_i) = F_i^{2cf}, \qquad i\in\{1,2\}.$$

Hence, applying Eq. (75), we have the error in requirement that can be achieved by $\boldsymbol{f}^r$ is

$$\begin{aligned} F_1^r &\geq R_1 - R_2\epsilon^{2cf}, \\ F_2^r &\geq R_2 - R_1\epsilon^{2cf}. \end{aligned} \tag{76}$$

As we set in the reduction that $\epsilon^{2cf} \leq \min\left\{\frac{\epsilon_1^r}{R_2}, \frac{\epsilon_2^r}{R_1}\right\}$, then we have

$$\tau_1^r \leq R_2\frac{\epsilon_1^r}{R_2} = \epsilon_1^r, \qquad \tau_2^r \leq R_1\frac{\epsilon_2^r}{R_1} = \epsilon_2^r.$$

Thus, we can conclude that if $\boldsymbol{f}^{2cf}$ is a solution to the 2CFA instance with the error ϵ^{2cf} as set in the reduction, we can map it back to a solution $\boldsymbol{f}^r$ to the 2CFRA instance with its error target achieved. □

5 Main Theorem

Now, we are ready to prove the main theorem.

Theorem 5.1 (Restatement of Theorem 3.1.)**.** *Given an* LPA *instance* $(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c},K,R,\epsilon^{lp})$ *where* $\boldsymbol{A}\in\mathbb{Z}^{m\times n},\boldsymbol{b}\in Z^m,\boldsymbol{c}\in\mathbb{Z}^n,K\in\mathbb{Z}$*, we can construct, in time* $O(\mathrm{nnz}(\boldsymbol{A})\log X)$ *where* $X = X(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c},K)$*, a* 2CFA *instance* $(G^{2cf},\boldsymbol{u}^{2cf},s_1,t_1,s_2,t_2,R^{2cf},\epsilon^{2cf})$ *such that*

$$|V^{2cf}|, |E^{2cf}| \leq 10^6\,\mathrm{nnz}(\boldsymbol{A})(3+\log X),$$

$$\left\|\boldsymbol{u}^{2cf}\right\|_{max}, R^{2cf} \leq 10^8\,\mathrm{nnz}^3(\boldsymbol{A})RX^2(2+\log X)^2,$$

and

$$\epsilon^{2cf} \geq \frac{\epsilon^{lp}}{10^{26}\,\mathrm{nnz}^{10}(\boldsymbol{A})R^3X^7(3+\log X)^7}.$$

If the LP *instance* $(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c},K,R)$ *has a solution, then the* 2CF *instance* $(G^{2cf},\boldsymbol{u}^{2cf},s_1,t_1,s_2,t_2,R^{2cf})$ *has a solution. Furthermore, if* $\boldsymbol{f}^{2cf}$ *is a solution to the* 2CFA *instance, then in time* $O(\mathrm{nnz}(\boldsymbol{A})\log X)$*, we can compute a solution* $\boldsymbol{x}$ *to the* LPA *instance.*

Proof. The theorem can be proved by putting all lemmas related to approximate problems in Section 4 together. Given an LPA instance $(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c},K,R,\epsilon^{lp})$, for each problem along the reduction chain, we bound its problem size, problem error, reduction time, and solution mapping time, in terms of the input parameters of the LPA instance.

1. LPA (by Lemma 4.3)
   - problem size: $n, m, \mathrm{nnz}(\boldsymbol{A}), R, X = X(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c},K)$
   - problem error: ϵ^{lp}
   - reduction time: \

- solution mapping time[9]: $O(n)$

2. LENA (by Lemma 4.3 and Lemma 4.6)
   Note: for simplification, in the following calculations, we replace n, m with $\operatorname{nnz}(\boldsymbol{A})$ since wlog we can assume $1 \leq n, m \leq \operatorname{nnz}(\boldsymbol{A})$.

   - problem size:

     $$\tilde{n} = 3\operatorname{nnz}(\boldsymbol{A}),\ \ \tilde{m} = 2\operatorname{nnz}(\boldsymbol{A}),\ \ \operatorname{nnz}(\tilde{\boldsymbol{A}}) \leq 4\operatorname{nnz}(\boldsymbol{A}),\ \ \tilde{R} \leq 10\operatorname{nnz}(\boldsymbol{A})RX,\ \ X(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}) = X$$

   - problem error: $\epsilon^{le} = \epsilon^{lp}$
   - reduction time: $O(\operatorname{nnz}(\boldsymbol{A}))$
   - solution mapping time: $O(\operatorname{nnz}(\boldsymbol{A}))$

3. 2-LENA (by Lemma 4.6 and Lemma 4.8)

   - problem size:

     $$\bar{n} \leq 8\operatorname{nnz}(\boldsymbol{A})(2+\log X),\ \ \bar{m} \leq 6\operatorname{nnz}(\boldsymbol{A})(1+\log X),\ \ \operatorname{nnz}(\bar{\boldsymbol{A}}) \leq 68\operatorname{nnz}(\boldsymbol{A})(1+\log X),$$
     $$\bar{R} \leq 160\operatorname{nnz}^2(\boldsymbol{A})RX^2(1+\log X),\ \ X(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}}) = 20\operatorname{nnz}(\boldsymbol{A})RX^2$$

   - problem error:

     $$\epsilon^{2le} = \frac{\epsilon^{lp}}{2X}$$

   - reduction time: $O(\operatorname{nnz}(\boldsymbol{A})\log X)$
   - solution mapping time: $O(\operatorname{nnz}(\boldsymbol{A})\log X)$

4. 1-LENA (by Lemma 4.8 and Lemma 4.11)

   - problem size:

     $$\hat{n} \leq 16\operatorname{nnz}(\boldsymbol{A})(2+\log X),\ \ \hat{m} \leq 14\operatorname{nnz}(\boldsymbol{A})(2+\log X),\ \ \operatorname{nnz}(\hat{\boldsymbol{A}}) \leq 272\operatorname{nnz}(\boldsymbol{A})(1+\log X),$$
     $$\hat{R} \leq 320\operatorname{nnz}^2(\boldsymbol{A})RX^2(1+\log X),\ \ X(\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}) = 20\operatorname{nnz}(\boldsymbol{A})RX^2$$

   - problem error:

     $$\epsilon^{1le} = \frac{\epsilon^{lp}}{16\operatorname{nnz}(\boldsymbol{A})X(3+\log X)}$$

   - reduction time: $O(\operatorname{nnz}(\boldsymbol{A})\log X)$
   - solution mapping time: $O(\operatorname{nnz}(\boldsymbol{A})\log X)$

5. FHFA (by Lemma 4.11 and Lemma 4.14)

   - problem size:

     $$\begin{aligned} |V^h| &\leq 28\operatorname{nnz}(\boldsymbol{A})(3+\log X), \\ |E^h| &\leq 1088\operatorname{nnz}(\boldsymbol{A})(1+\log X), \\ |F^h| &\leq 14\operatorname{nnz}(\boldsymbol{A})(2+\log X), \\ h &\leq 30\operatorname{nnz}(\boldsymbol{A})(2+\log X), \\ \left\|\boldsymbol{u}^h\right\|_{max} &\leq 320\operatorname{nnz}^2(\boldsymbol{A})RX^2(1+\log X) \end{aligned}$$

[9] The time needed to turn a solution to the next problem in the reduction chain to a solution to this problem.

- problem error:

$$\epsilon_l^h = \frac{\epsilon^{lp}}{960\,\mathrm{nnz}^2(\boldsymbol{A})RX^3(3+\log X)},$$
$$\epsilon_u^h = \frac{\epsilon^{lp}}{960\,\mathrm{nnz}^2(\boldsymbol{A})RX^3(3+\log X)},$$
$$\epsilon_d^h = \frac{\epsilon^{lp}}{96\,\mathrm{nnz}(\boldsymbol{A})X(3+\log X)},$$
$$\epsilon_h^h = \frac{\epsilon^{lp}}{3\cdot 10^5\,\mathrm{nnz}^4(\boldsymbol{A})RX^3(3+\log X)^3}$$

- reduction time: $O(\mathrm{nnz}(\boldsymbol{A})\log X)$
- solution mapping time: $O(\mathrm{nnz}(\boldsymbol{A})\log X)$

6. FPHFA (by Lemma 4.14 and Lemma 4.17)

- problem size:

$$|V^p| \le 1116\,\mathrm{nnz}(\boldsymbol{A})(3+\log X),$$
$$|E^p| \le 2176\,\mathrm{nnz}(\boldsymbol{A})(1+\log X),$$
$$|F^p| \le 14\,\mathrm{nnz}(\boldsymbol{A})(2+\log X),$$
$$p \le 1088\,\mathrm{nnz}(\boldsymbol{A})(1+\log X),$$
$$\|\boldsymbol{u}^p\|_{max} \le 320\,\mathrm{nnz}^2(\boldsymbol{A})RX^2(1+\log X)$$

- problem error:

$$\epsilon_l^p = \frac{\epsilon^{lp}}{960\,\mathrm{nnz}^2(\boldsymbol{A})RX^3(3+\log X)},$$
$$\epsilon_u^p = \frac{\epsilon^{lp}}{960\,\mathrm{nnz}^2(\boldsymbol{A})RX^3(3+\log X)},$$
$$\epsilon_d^p = \frac{\epsilon^{lp}}{2\cdot 10^5\,\mathrm{nnz}^2(\boldsymbol{A})X(3+\log X)^2},$$
$$\epsilon_h^p = \frac{\epsilon^{lp}}{3\cdot 10^5\,\mathrm{nnz}^4(\boldsymbol{A})RX^3(3+\log X)^3}$$

- reduction time: $O(\mathrm{nnz}(\boldsymbol{A})\log X)$
- solution mapping time: $O(\mathrm{nnz}(\boldsymbol{A})\log X)$

7. SFFA (by Lemma 4.17 and Lemma 4.20)

- problem size:

$$|V^s| \le 5468\,\mathrm{nnz}(\boldsymbol{A})(3+\log X),$$
$$|E^s| \le 9792\,\mathrm{nnz}(\boldsymbol{A})(1+\log X),$$
$$|F^s| \le 2190\,\mathrm{nnz}(\boldsymbol{A})(2+\log X),$$
$$|S_1| \le 4352\,\mathrm{nnz}(\boldsymbol{A})(1+\log X),$$
$$|S_2| \le 3264\,\mathrm{nnz}(\boldsymbol{A})(1+\log X),$$
$$\|\boldsymbol{u}^s\|_{max} \le 320\,\mathrm{nnz}^2(\boldsymbol{A})RX^2(1+\log X)$$

- problem error:

$$\begin{aligned}
\epsilon_l^s, \epsilon_u^s &= \frac{\epsilon^{lp}}{2 \cdot 10^9 \operatorname{nnz}^6(\boldsymbol{A}) R^2 X^5 (3 + \log X)^4}, \\
\epsilon_{d1}^s, \epsilon_{t2}^s &= \min \left\{ \frac{\epsilon^{lp}}{3 \cdot 10^9 \operatorname{nnz}^3(\boldsymbol{A}) X (3 + \log X)^3}, \frac{\epsilon^{lp}}{3 \cdot 10^6 \operatorname{nnz}^4(\boldsymbol{A}) R X^3 (3 + \log X)^3} \right\} \\
&\geq \frac{\epsilon^{lp}}{3 \cdot 10^9 \operatorname{nnz}^4(\boldsymbol{A}) R X^3 (3 + \log X)^3} \\
\epsilon_{d2}^s, \epsilon_{t1}^s &= \frac{\epsilon^{lp}}{3 \cdot 10^6 \operatorname{nnz}^4(\boldsymbol{A}) R X^3 (3 + \log X)^3}
\end{aligned}$$

- reduction time: $O(\operatorname{nnz}(\boldsymbol{A}) \log X)$
- solution mapping time: $O(\operatorname{nnz}(\boldsymbol{A}) \log X)$

8. 2CFFA (by Lemma 4.20 and Lemma 4.23)

- problem size:

$$\begin{aligned}
|V^f| &\leq 20700 \operatorname{nnz}(\boldsymbol{A})(3 + \log X), \\
|E^f| &\leq 40256 \operatorname{nnz}(\boldsymbol{A})(1 + \log X), \\
|F^f| &\leq 39224 \operatorname{nnz}(\boldsymbol{A})(2 + \log X), \\
\left\| \boldsymbol{u}^f \right\|_{max} &\leq 320 \operatorname{nnz}^2(\boldsymbol{A}) R X^2 (1 + \log X)
\end{aligned}$$

- problem error:

$$\begin{aligned}
\epsilon_l^f, \epsilon_u^f &= \frac{\epsilon^{lp}}{10^{17} \operatorname{nnz}^7(\boldsymbol{A}) R^2 X^5 (3 + \log X)^5}, \\
\epsilon_{d1}^f, \epsilon_{d2}^f &= \frac{\epsilon^{lp}}{3 \cdot 10^{14} \operatorname{nnz}^5(\boldsymbol{A}) R X^3 (3 + \log X)^4}
\end{aligned}$$

- reduction time: $O(\operatorname{nnz}(\boldsymbol{A}) \log X)$
- solution mapping time: $O(\operatorname{nnz}(\boldsymbol{A}) \log X)$

9. 2CFRA (by Lemma 4.23 and Lemma 4.26)

- problem size:

$$\begin{aligned}
|V^r| &\leq 2 \cdot 10^5 \operatorname{nnz}(\boldsymbol{A})(3 + \log X), \\
|E^r| &\leq 3 \cdot 10^5 \operatorname{nnz}(\boldsymbol{A})(3 + \log X), \\
\|\boldsymbol{u}^r\|_{max} \leq M^f &\leq |E^f| \left\| \boldsymbol{u}^f \right\|_{max} \leq 2 \cdot 10^7 \operatorname{nnz}^3(\boldsymbol{A}) R X^2 (1 + \log X)^2, \\
R_1, R_2 &\leq 4 \cdot 10^7 \operatorname{nnz}^3(\boldsymbol{A}) R X^2 (1 + \log X)^2
\end{aligned}$$

- problem error:

$$\begin{aligned}
\epsilon^r &= \frac{\epsilon^{lp}}{3 \cdot 10^{25} \operatorname{nnz}^{10}(\boldsymbol{A}) R^3 X^7 (3 + \log X)^7} \\
\epsilon_1^r, \epsilon_2^r &= \frac{\epsilon^{lp}}{3 \cdot 10^{17} \operatorname{nnz}^7(\boldsymbol{A}) R^2 X^5 (3 + \log X)^5}
\end{aligned}$$

- reduction time: $O(\text{nnz}(\boldsymbol{A}) \log X)$
- solution mapping time: $O(\text{nnz}(\boldsymbol{A}) \log X)$

10. 2CFA (by Lemma 4.26)

- problem size:

$$|V^{2cf}|, |E^{2cf}| \leq 10^6 \,\text{nnz}(\boldsymbol{A})(3 + \log X),$$
$$\left\|\boldsymbol{u}^{2cf}\right\|_{max}, R^{2cf} \leq 10^8 \,\text{nnz}^3(\boldsymbol{A}) R X^2 (2 + \log X)^2$$

- problem error:

$$\begin{aligned}\epsilon^{2cf} &= \min\left\{\frac{\epsilon^{lp}}{3 \cdot 10^{25} \,\text{nnz}^{10}(\boldsymbol{A}) R^3 X^7 (3 + \log X)^7}, \frac{\epsilon^{lp}}{2 \cdot 10^{25} \,\text{nnz}^{10}(\boldsymbol{A}) R^3 X^7 (3 + \log X)^7}\right\} \\ &\geq \frac{\epsilon^{lp}}{10^{26} \,\text{nnz}^{10}(\boldsymbol{A}) R^3 X^7 (3 + \log X)^7}\end{aligned}$$

- reduction time: $O(\text{nnz}(\boldsymbol{A}) \log X)$
- solution mapping time: \

□

By applying Theorem 5.1, we can prove Corollary 3.2, which states that if there exists a fast high-accuracy 2CFA solver, then there also exists an almost equally fast high-accuracy LPA solver.

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A 2CFA simplification

In this Appendix, we prove Lemma 2.7 for completeness. We restate the lemma in the following.

Lemma A.1 (Restatement of Lemma 2.7: 2CFA simplification)**.** *Given a solution to a* 2CFA *instance* $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2, R, \epsilon, \epsilon')$*, we can reduce it, in linear time, to a solution to* 2CFA $(G, \boldsymbol{u}, s_1, t_1, s_2, t_2, R, \tilde{\epsilon}, 0)$*, where* $\tilde{\epsilon}$ *is the error in congestion and satisfies*

$$\tilde{\epsilon} = \frac{\epsilon + \epsilon'}{1 - \epsilon'}.$$

Proof. By the definition of error in demand, we have

$$F_1 + F_2 \geq (1 - \epsilon')R.$$

We scale $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ by $\frac{1}{1-\epsilon'}$, which takes linear time, and get

$$\tilde{\boldsymbol{f}}_i = \frac{\boldsymbol{f}_i}{1 - \epsilon'}, \qquad i = \{1, 2\}.$$

Then, there is no error in demand for $\tilde{\boldsymbol{f}}_1$ and $\tilde{\boldsymbol{f}}_2$ since

$$\tilde{F}_1 + \tilde{F}_2 \geq \frac{1}{1 - \epsilon'}(1 - \epsilon')R = R.$$

And $\tilde{\boldsymbol{f}}_1$ and $\tilde{\boldsymbol{f}}_2$ are only with $\tilde{\epsilon}$ error in congestion because it satisfies:

1. direction constraint: $\tilde{\boldsymbol{f}}_1, \tilde{\boldsymbol{f}}_2 \geq \boldsymbol{0}$;
2. capacity constraint with error in congestion:

$$\tilde{\boldsymbol{f}}_1 + \tilde{\boldsymbol{f}}_2 \leq \frac{1}{1 - \epsilon'}(1 + \epsilon)\boldsymbol{u} = \left(1 + \frac{\epsilon + \epsilon'}{1 - \epsilon'}\right)\boldsymbol{u} := (1 + \tilde{\epsilon})\boldsymbol{u};$$

3. conservation of flows: $\sum_{u:(u,v)\in E} \tilde{\boldsymbol{f}}_i(u, v) = \sum_{w:(v,w)\in E} \tilde{\boldsymbol{f}}_i(v, w),\ \forall v \in V \backslash \{s_i, t_i\}, i \in \{1, 2\}.$

□