AM 221: Advanced Optimization Spring 2018

Prof. Rasmus Kyng *Section 3 — Friday, Feb. 9th*

1. Equivalent Formulation of Farkas' lemma. Remember Farkas' lemma as stated in class: let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$ then exactly one of the following two statements holds:

1. $\exists \mathbf{x} \in \mathbb{R}^n,\ A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$.

2. $\exists \mathbf{p} \in \mathbb{R}^m,\ \mathbf{p}^\intercal A \geq 0$ and $\mathbf{p}^\intercal \mathbf{b} < 0$.

We show this is equivalent to the following reformulation: exactly one of the following two statements holds:

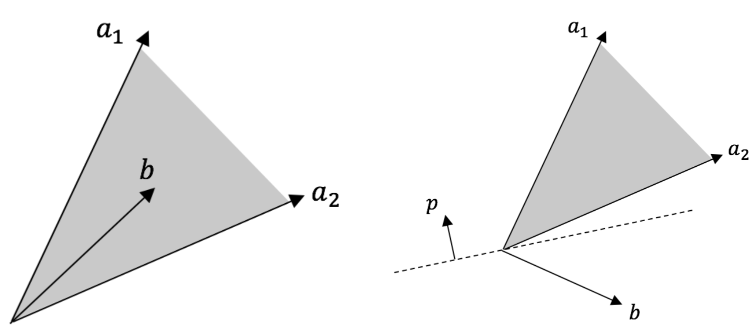
1. $\exists \mathbf{x} \in \mathbb{R}^n,\ A\mathbf{x} \leq \mathbf{b}$,

2. $\exists \mathbf{p} \in \mathbb{R}^m,\ \mathbf{p} \geq 0,$ and $\mathbf{p}^\intercal A = 0$ and $\mathbf{p}^\intercal \mathbf{b} < 0$.

Letting $A' = \begin{bmatrix} A & -A & I \end{bmatrix}^T$ and $\mathbf{b}' = \begin{bmatrix} \mathbf{b} & -\mathbf{b} & \underline{0} \end{bmatrix}^T$, we note that $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} \geq 0$ can be written as $A'\mathbf{x} \leq \mathbf{b}'$. This system is infeasible iff $\exists \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{R}^m$, $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \geq 0$ such that $(\mathbf{p}_1 - \mathbf{p}_2)^T A = \mathbf{p}_3^T$ and $(\mathbf{p}_1 - \mathbf{p}_2)^T \mathbf{b} < 0$. By defining $\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2$, we see that $\mathbf{p}^T A \geq 0$ and $\mathbf{p}^T \mathbf{b} < 0$.

Geometric interpretation of Farkas' lemma:

Starting with $A \in \mathbb{R}^{m \times n}$ with columns a_i, the theorem states that a vector $\mathbf{b}$ is either inside the convex cone generated by the columns of A or is outside. When inside, the first condition states that $\mathbf{b} = \sum_{i=1}^n x_i a_i$, and $x_i \geq 0$ for $i = 1, ..., n$, illustrated in figure (a). When outside, the second condition states the existance of a vector $\mathbf{p}$ normal to the hyperplane separating $\mathbf{b}$ from the convex cone, illustrated in figure (b). More formally, this is stated as $\mathbf{p}^T a_i \geq 0$ for $i = 1, ..., m$ and $\mathbf{p}^T b < 0$.



(a) Condition 1 (b) Condition 2

2. Separation of Polyhedra. Let us consider two polyhedra, $P = \{\mathbf{x} \in \mathbb{R}^n \,|\, A\mathbf{x} \leq \mathbf{b}\}$ and $Q = \{\mathbf{x} \in \mathbb{R}^n \,|\, C\mathbf{x} \leq \mathbf{d}\}$, for $A, B \in \mathbb{R}^{m \times n}$ and $\mathbf{b}, \mathbf{d} \in \mathbb{R}^m$.

a. Write a linear program which finds $\mathbf{x} \in P \cap Q$ if $P \cap Q \neq \emptyset$ and which is infeasible when $P \cap Q = \emptyset$.

$$\begin{aligned} \max_{\mathbf{x} \in \mathbb{R}^n} & \ 0^T \mathbf{x} \\ \text{s.t } & A\mathbf{x} \leq \mathbf{b} \\ & C\mathbf{x} \leq \mathbf{d} \end{aligned}$$

b. Write the dual of the program found in part a.

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^n} & \ \mathbf{p}_1^T \mathbf{b} + \mathbf{p}_2^T \mathbf{d} \\ \text{s.t } & \mathbf{p}_1^T A + \mathbf{p}_2^T C = 0^T \\ & \mathbf{p}_1, \mathbf{p}_2 \geq 0 \end{aligned}$$

c. Show that the polyhedra P and Q have an empty intersection if and only if there exists a hyperplane which separates them strictly, *i.e.* there exists $\mathbf{c} \in \mathbb{R}^n$ such that:

$$\mathbf{c}^\mathsf{T} \mathbf{x} < \mathbf{c}^\mathsf{T} \mathbf{y}, \quad \mathbf{x} \in P, \ \mathbf{y} \in Q$$

Recall that if the primal is infeasible, the dual is either infeasible or unbounded. We have that $\mathbf{p}_1 = 0$, $\mathbf{p}_2 = 0$ is feasible for the dual, so it must be unbounded. We can then say that for every $z \in \mathbb{R}$, $\exists \mathbf{p}_1, \mathbf{p}_2 \geq 0$ such that $\mathbf{p}_1^T \mathbf{b} + \mathbf{p}_2^T \mathbf{d} \leq z$. We fix an $\epsilon > 0$ and take $z = -\epsilon$ so that $\mathbf{p}_1^T \mathbf{b} + \mathbf{p}_2^T \mathbf{d} < 0$.

By definition of P and Q we have that for all $\mathbf{x} \in P$ and $\mathbf{y} \in Q$,

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ C\mathbf{y} &\leq \mathbf{d} \end{aligned}$$

Multiplying by $\mathbf{p}_1^T$ and $\mathbf{p}_2^T$ on both sides, we get:

$$\begin{aligned} \mathbf{p}_1^T A\mathbf{x} &\leq \mathbf{p}_1^T \mathbf{b} \\ \mathbf{p}_2^T C\mathbf{y} &\leq \mathbf{p}_2^T \mathbf{d} \end{aligned}$$

Note that we have the dual constraint that $\mathbf{p}_1^T A + \mathbf{p}_2^T C = 0^T$, which we can rearrange as $\mathbf{p}_1^T A = -\mathbf{p}_2^T C$. Substituting this in the second constraint yields:

$$(-\mathbf{p}_1^T A) C\mathbf{y} \leq \mathbf{p}_2^T \mathbf{d}$$

Add the constraints and note that from our choice in z,

$$\begin{aligned} \mathbf{p}_1^T A(\mathbf{x} - \mathbf{y}) \leq \mathbf{p}_1^T \mathbf{b} + \mathbf{p}_2^T \mathbf{d} &< 0 \\ \mathbf{p}_1^T A\mathbf{x} \leq \mathbf{p}_1^T \mathbf{b} < \mathbf{p}_2^T \mathbf{d} &\leq \mathbf{p}_1^T A\mathbf{y} \end{aligned}$$

We can see that setting $c = (\mathbf{p}_1^T A)^T$ satisfies

$$\mathbf{c}^\mathsf{T} \mathbf{x} < \mathbf{c}^\mathsf{T} \mathbf{y}, \quad \mathbf{x} \in P, \ \mathbf{y} \in Q$$