AM 221: Advanced Optimization	Spring 2016
Prof. Yaron Singer	*Section 2 — Wednesday, Feb. 3rd*

1. Characterization of Convex Sets through Secants

a. Show that the convex subsets of $\mathbb{R}$ are exactly the intervals.

b. In $\mathbb{R}^d$, the affine line going through $\mathbf{x}$ of direction $\mathbf{d}$ is the set:

$$L_{\mathbf{x},\mathbf{d}} = \{\mathbf{x} + \lambda\mathbf{d},\ \lambda \in \mathbb{R}\}$$

Show that for any $(\mathbf{x}, \mathbf{d}) \in \mathbb{R}^d \times \mathbb{R}^d$, $L_{\mathbf{x},\mathbf{d}}$ is a convex set.

c. Show that $C \subseteq \mathbb{R}^d$ is convex if and only if for all affine line D, $C \cap D$ is convex.

2. Graph of convex functions. Show that the epigraph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is convex. Recall that the epigraph of a function f is defined by:

$$E_f \overset{\text{def}}{=} \{(\mathbf{x}, \mathbf{y}) \,|\, \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{y} \geq f(\mathbf{x})\}$$

3. Supporting Hyperplanes. In this problem, we want to show the *supporting hyperplane theorem*: let $C \subset \mathbb{R}^n$ be a closed convex set and $\mathbf{x}_0$ be a point on the boundary of C, then there exists a supporting hyperplane of C containing $\mathbf{x}_0$, that is, there exists $\mathbf{a} \in \mathbb{R}^n$ such that:

$$\forall \mathbf{x} \in C, \quad \mathbf{a}^\intercal\mathbf{x} \leq \mathbf{a}^\intercal\mathbf{x}_0$$

a. Show that there exists a sequence $(\mathbf{z}_n)_{n\geq 0}$ of elements in $\mathbb{R}^n \setminus C$ which converges to $\mathbf{x}_0$.

b. Show that there exists a sequence $(\mathbf{a}_n)_{n\geq 0}$ in $B_2(0,1) \overset{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n \,|\, \|\mathbf{x}\|_2 \leq 1\}$ such that:

$$\forall \mathbf{x} \in C, \quad \mathbf{a}_n^\intercal\mathbf{x} \leq \mathbf{a}_n^\intercal\mathbf{z}_n$$

c. Conclude.

4. Canonical form, Standard form. Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. A linear programming problem (of optimization variable $\mathbf{x} \in \mathbb{R}^n$ is said to be in *canonical form* if its constraints have the following form:

$$A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$$

A linear programming problem is said to be in *standard form* if its constraints are written as:

$$A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$$

a. Show that any linear programming problem can be put in standard form or canonical form. In particular, it is possible to go from the canonical form to the standard form, and vice versa.

b. Consider the following linear optimization problem in canonical form:

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & 7x_1 + 6x_2 \\
s.t. \quad & 3x_1 + x_2 \leq 120 \\
& x_1 + 2x_2 \leq 160 \\
& x_1 \leq 35 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{aligned}
$$

Put the problem in standard form.

5. Equivalent Formulation of Farkas' lemma. Remember Farkas' lemma as stated in class: let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$ then exactly one of the following two statements holds:

1. $\exists \mathbf{x} \in \mathbb{R}^n,\ A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$.
2. $\exists \mathbf{p} \in \mathbb{R}^m,\ \mathbf{p}^\intercal A \leq 0$ and $\mathbf{p}^\intercal \mathbf{b} > 0$.

Show that this is equivalent to the following reformulation. Exactly one of the following two statements holds:

1. $\exists \mathbf{x} \in \mathbb{R}^n,\ A\mathbf{x} \leq \mathbf{b}$,
2. $\exists \mathbf{p} \in \mathbb{R}^m,\ \mathbf{p} \geq 0,$ and $\mathbf{p}^\intercal A = 0$ and $\mathbf{p}^\intercal \mathbf{b} < 0$.