AM 221: Advanced Optimization	Spring 2018
Dr. Rasmus Kyng	*Problem Set 4 — Due Thursday, Feb. 22th at 23:59*

Instructions: All your solutions should be prepared in LaTeX and the PDF and .tex should be submitted to Canvas. Please submit all your files as ONE archive of filetype zip, tgz, or tar.gz. For each question, a well-written and correct answer will be selected a sample solution for the entire class to enjoy. If you prefer that we do not use your solutions, please indicate this clearly on the first page of your assignment.

1. Simplex example Consider the following linear program:

$$\begin{aligned} \min\ & -x_1 + 2x_2 \\ \text{s.t. } & 3x_1 + x_2 + x_3 = 6 \\ & x_1 - 2x_2 + x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

a. Run the simplex algorithm on the linear program to find an optimal point. Start from the vertex $(x_1, x_2, x_3, x_4) = (0, 0, 6, 1)$. List the vertex visited at each round.

b. What is the optimal value of the primal program?

c. Write down the dual of the linear program.

d. State an optimal solution to the dual and explain why it proves that the simplex algorithm found an optimal soluton.

2. Max-Flow Min-Cut. In this problem we will study *flow networks*. A *flow network* is defined by a finite set of vertices V with two distinguished vertices, the *source* s and the *sink* t. Each edge $(u, v) \in V \times V$ has a *capacity* $c_{uv} \in \mathbb{R}^+$. Finally the source s has no incoming capacity, *i.e* $c_{us} = 0$ for $u \in V$ and the sink t has no outgoing capacity, *i.e* $c_{tu} = 0$ for $u \in V$.

A *flow* of the network is a family $(f_{uv})_{(u,v)\in V\times V}$ of real numbers satisfying the following constraints:

- *Positivity:* $f_{uv} \geq 0$ for all $(u, v) \in V \times V$.
- *Capacity constraint:* for all $(u, v) \in V \times V$, $f_{uv} \leq c_{uv}$.
- *Flow conservation:* for all $u \in V \setminus \{s, t\}$, $\sum_{v\in V} f_{vu} = \sum_{v\in V} f_{uv}$.

The amount of flow *leaving* s is $\sum_{u\in V} f_{su}$ and is called the *value of the flow* f. The goal of the *maximum flow problem* is to find a flow f satisfying the three conditions above whose value is maximal.

a. Show that the value of the flow f is equal to $\sum_{v \in V} f_{vt}$. Hence, the value of the flow could equivalently be defined as the amount of flow entering t.

b. Write the maximum flow problem as a linear program. Is this LP feasible? Is it bounded?

An $s-t$ *cut* of G is a partition of V into two sets: $V = S \cup T$ with $S \cap T = \emptyset$, such that $s \in S$ and $t \in T$. The *cost* c of an $s-t$ cut is defined by $c(S,T) := \sum_{(u,v) \in S \times T} c_{uv}$. The goal of the *minimum cut problem* is to find an $s-t$ cut of minimal cost.

d. Formulate a 0/1-integer program for the minimum cut problem (recall that 0/1-integer programs and their relaxations where covered in Section 1). Let us name this problem MinimumCut. Write the LP relaxation of MinimumCut. We will name this relaxation MinimumCut-LP.

e. Derive the dual of the maximum flow LP that you wrote in b. and show that it is equivalent to MinimumCut-LP. Show that the optimal value of MinimumCut is an upper bound on the value of the optimal flow.

f. Assuming that MinimumCut and MinimumCut-LP have the same optimal value [**bonus credits** if you prove this fact], prove the famous *max-flow min-cut theorem*:

The maximum value of a flow is equal to the minimal cost of an $s-t$ cut.

3. Optimality criteria. Let f be a differentiable convex function from $\mathbb{R}^n$ to $\mathbb{R}$. We consider the following optimization problem:

$$\min_{\mathbf{x} \in C} f(\mathbf{x})$$

where $C \subseteq \mathbb{R}^n$ is a closed convex set.

a. Show that $\mathbf{x}^* \in C$ is an optimal solution to the above problem if and only if:

$$\forall \mathbf{y} \in C,\ \nabla f(\mathbf{x}^*)^\intercal (\mathbf{y} - \mathbf{x}^*) \geq 0 \tag{P}$$

b. Show that when $\nabla f(\mathbf{x}^*) \neq 0$ this implies that $\mathbf{x}^*$ lies on the boundary of C and that $\nabla f(\mathbf{x}^*)$ defines a supporting hyperplane of C at $\mathbf{x}^*$. Remember that the boundary $\delta(C)$ of a closed set C is defined by:

$$\delta(C) \stackrel{\text{def}}{=} \{\mathbf{x} \in C \,|\, \forall r > 0, B(\mathbf{x}, r) \not\subseteq C\}$$

where $B(\mathbf{x}, r)$ denotes the ℓ_2-ball of center $\mathbf{x}$ and radius r.

c. Show that when $C = \mathbb{R}^n$, $\mathbf{x}^* \in \mathbb{R}^n$ satisfies condition (P) of part a. iff:

$$\nabla f(\mathbf{x}^*) = 0$$

Then observe (you don't have to prove it) that by combining part a. and c. we obtain the following theorem.

Theorem 1. $\mathbf{x}^* \in \mathbb{R}^n$ *is an optimal solution to the problem:*

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$$

if and only if $\nabla f(\mathbf{x}^*) = 0$.